

Embeddings of non-simply-connected 4-manifolds in 7-space

I. Classification modulo knots

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Abstract

We work in the smooth category. Let N be a closed connected orientable 4-manifold with torsion free H_1 , where $H_q := H_q(N; \mathbb{Z})$. Our main result is a *complete readily calculable classification of embeddings* $N \rightarrow \mathbb{R}^7$, up to the equivalence relation generated by isotopy and embedded connected sum with embeddings $S^4 \rightarrow \mathbb{R}^7$. Such a classification was already known when $H_1 = 0$ by the work of Bo  chat, Haefliger and Hudson from 1970. Our results for $H_1 \neq 0$ are new. The classification involves the Bo  chat-Haefliger invariant $\varkappa(f) \in H_2$, and two new invariants: a Seifert bilinear form $\lambda(f) : H_3 \times H_3 \rightarrow \mathbb{Z}$ and β -invariant $\beta(f)$ which assumes values in a quotient of H_1 depending on the values of $\varkappa(f)$ and $\lambda(f)$.

For $N = S^1 \times S^3$ we give a geometrically defined 1–1 correspondence between the set of equivalence classes of embeddings and an explicit quotient of the set $\mathbb{Z} \oplus \mathbb{Z}$.

Our proof is based on Kreck’s modified surgery approach to the classification of embeddings, and also uses parametric connected sum.

1 Introduction and main results

1.1 Overview

In this paper we consider *smooth*¹ manifolds, embeddings and isotopies and fix the following notation:

- $S^m \subset \mathbb{R}^{m+1}$ is the unit m -sphere;
- N is a closed connected orientable 4-manifold;
- $[f]$ denotes the isotopy class of an embedding $f : N \rightarrow S^m$;
- $E^m(N)$ denotes the set of isotopy classes of embeddings $f : N \rightarrow S^m$.²

By a classification of $E^m(N)$ we mean a *complete, readily calculable* classification of this set.³

Embeddings of N into S^m were classified for $m \geq 9$ by Whitney-Wu, for $m = 8$ by Haefliger and Hirsch, and for $N = S^4$ and $m = 7$ by Haefliger, giving

$$|E^m(N)| = 1 \quad \text{for } m \geq 9, \quad E^8(N) = H_1(N; \mathbb{Z}_2), \quad E^7(S^4) \cong \mathbb{Z}_{12}.$$

Here the equality sign between sets denotes the existence of a ‘geometrically defined’ bijection, and the isomorphism is a group isomorphism for the group structure defined below.⁴

*Supported in part by the Russian Foundation for Basic Research Grant No. 15-01-06302, by Simons-IUM Fellowship and by the D. Zimin Dynasty Foundation.

¹In this paper ‘smooth’ means ‘ C^1 -smooth’. Cf. [Sk15, footnote 2].

²For $m \geq n + 2$ the classifications of embeddings of n -manifolds into S^m and into \mathbb{R}^m are the same [MAH, §1]. It is technically more convenient to consider embeddings into S^m instead of \mathbb{R}^m .

³For a discussion of the adjective ‘readily calculable’ see [Sk10, footnote 1], [MAH, §1].

⁴For more information and references see [MAM]. For results on embeddings of n -manifolds into \mathbb{R}^{2n-1} see [Ya84, Sa99, Sk10’, To10]. For a review of the general Knotting Problem see §1.3.

The group structure on $E^7(S^4)$ and its action on $E^7(N)$ are defined as follows. Represent elements of $E^7(N)$ and of $E^7(S^4)$ by embeddings $f : N \rightarrow S^7$ and $g : S^4 \rightarrow S^7$ whose images are contained in disjoint balls. Join the images of f, g by an arc whose interior misses the images. Let $[f] \# [g]$ be the isotopy class of the *embedded connected sum* of f and g along this arc, cf. [Ha66, Theorem 1.7], [Av16, §1]. The isotopy class of class of the embedded connected sum depends only on the the isotopy classes $[f]$ and $[g]$.⁵ Define the operation

$$\# : E^7(N) \times E^7(S^4) \rightarrow E^7(N) \quad \text{by} \quad ([f], [g]) \mapsto [f] \# [g].$$

When $N = S^4$, $\#$ defines a group structure on $E^7(S^4)$ [Ha66]. Clearly $\#$ is an action of $E^7(S^4)$ on the set $E^7(N)$. We define

$$E_{\#}^7(N) := E^7(N) / E^7(S^4)$$

to be the quotient of this action and by $q_{\#} : E^7(N) \rightarrow E_{\#}^7(N)$ the quotient map.

In [BH70] Boéchat and Haefliger classified $E_{\#}^7(N)$ when $H_1(N; \mathbb{Z}) = 0$. The action of the knots was investigated in [Sk10] and determined when $H_1(N; \mathbb{Z}) = 0$ in [CS11], which also classified $E^7(N)$ in this case.

In this paper we classify $E_{\#}^7(N)$ when $H_1(N; \mathbb{Z})$ is torsion free; see Theorems 1.1 and 1.2 below. This requires finding a complete set of invariants and constructing embeddings realizing particular values of these invariants. Lemmas 1.3, §2.2, §2.3 describe the invariants we use and §1.4 gives an overview of the proof of their completeness. The beginning of §1.2 gives explicit construction of embeddings $S^1 \times S^3 \rightarrow S^7$. For general N , we use a parametric connected sum operation on embeddings which is described in §2.4. We create new embeddings $N \rightarrow S^7$ from a fixed embedding $f_0 : N \rightarrow S^7$ using parametric connected sum with embeddings $S^1 \times S^3 \rightarrow S^7$. Consequently, embeddings of $S^1 \times S^3$ play a key role in the classification of embeddings for all N .

In later work [CSII, CSIII] we extend methods of this paper and give the classification, under the same ‘torsion free’ condition, of $E^7(N)$ (up to an indeterminacy in certain cases) and of the *piecewise linear* (PL) isotopy classes. Some parts of those results easily follow from this paper, but we state those parts in [CSII, CSIII] not here.

The remaining subsections of §1 are written so that they can be read independently.

1.2 Main results

We first define a family of embeddings $\tau_{\alpha} : S^1 \times S^3 \rightarrow S^7$ and a corresponding map

$$\tau : \mathbb{Z}^2 \rightarrow E^7(S^1 \times S^3).$$

Let $V_{4,2}$ denote the Stiefel manifold of orthonormal 2-frames in \mathbb{R}^4 . Take a smooth map $\alpha : S^3 \rightarrow V_{4,2}$. Regarding $V_{4,2} \subset (\mathbb{R}^4)^2$, write $\alpha(x) = (\alpha_1(x), \alpha_2(x))$. Define the adjunction map $\mathbb{R}^2 \times S^3 \rightarrow \mathbb{R}^4$ by $((s, t), x) \mapsto \alpha_1(x)s + \alpha_2(x)t$. (Regarding $V_{4,2} \subset (\mathbb{R}^4)^{\mathbb{R}^2}$, this map is obtained from α by the exponential law.) Denote by $\bar{\alpha} : S^1 \times S^3 \rightarrow S^3$ the restriction of the adjunction map. We define the embedding τ_{α} to be the composition

$$S^1 \times S^3 \xrightarrow{\bar{\alpha} \times \text{pr}_2} S^3 \times S^3 \xrightarrow{i} S^7, \quad \text{where} \quad i(x, y) := (y, x) / \sqrt{2} \quad \text{and} \quad \text{pr}_2(x, y) = y.$$

We define the map τ by $\tau(l, b) := [\tau_{\alpha}]$, where $\alpha : S^3 \rightarrow V_{4,2}$ represents $(l, b) \in \pi_3(V_{4,2})$ (for the standard identification $\pi_3(V_{4,2}) = \mathbb{Z}^2$ described in §2.1).⁶

We define $\tau_{\#} := q_{\#} \tau$.

⁵This is proved analogously to the case $X = D_+^0$ of [Sk15, Standardization Lemma 2.1.b], cf. [Sk15, Remark 2.3.a], because the construction of $\#$ has an analogue for isotopy.

⁶We give an alternative construction of $\tau(0, 1)$ and $\tau(1, 0)$ in Remark 2.22, cf. Lemma 2.21. For other constructions see [MAM, Examples of knotted tori].

Theorem 1.1. *The map*

$$\tau_{\#} : \mathbb{Z}^2 \rightarrow E_{\#}^7(S^1 \times S^3)$$

is a surjection such that

$$\tau_{\#}(l, b) = \tau_{\#}(l', b') \iff (l = l' \text{ and } b \equiv b' \pmod{2l}).$$

Before stating our main results for the general case, we establish some conventions, notation and definitions.

Convention on coefficients. Unless otherwise stated, we omit \mathbb{Z} -coefficients from the notation of (co)homology groups. We identify the coefficient group \mathbb{Z}_d with $H_0(X; \mathbb{Z}_d)$, the zero-dimensional homology group of a connected oriented manifold X .

Notation for characteristic classes and intersections in manifolds. Let $H_q := H_q(N)$. We denote the Poincaré dual of a characteristic class by adding a superscript ‘ $*$ ’, so for example $w_2^*(N) \in H_2(N; \mathbb{Z}_2)$ is the Poincaré dual of the second Stiefel-Whitney class. The homology intersection products in an n -manifold M are denoted by \cap_M :

$$\cap_M : H_i(M) \times H_j(M) \rightarrow H_{n-i-j}(M).$$

The well-known definitions of such products are recalled in [Sk10, Remark 2.3]. Let $\sigma(N)$ be the signature of the intersection form $H_2 \times H_2 \rightarrow \mathbb{Z}$. For the intersection *powers* we omit subscripts indicating the manifold M , so, for example, x^2 denotes $x \cap_M x$. Let ρ_n be the reduction modulo n . The intersection $x \cap_M y$ of a \mathbb{Z} -homology class x and a \mathbb{Z}_n -homology class y is defined as the \mathbb{Z}_n -homology class $\rho_n x \cap_M y$.

If $H_1 = 0$, then the Boéchat-Haeffliger invariant (defined in §2.2)

$$\varkappa_{\#} : E_{\#}^7(N) \rightarrow H_2^{DIFF} := \{u \in H_2 \mid \rho_2 u = w_2^*(N), u \cap_N u = \sigma(N)\} \subset H_2$$

*is a 1–1 map.*⁷

Our second main result is a generalization of this statement to non-simply-connected 4-manifolds.

Definition of div , \bar{l} and a symmetric pair. For an element u of a free abelian group denote by $\text{div } u$ the divisibility of u , i.e. $\text{div } 0 = 0$ and $\text{div } u$ is the largest integer which divides u for $u \neq 0$. For an element u of an abelian group G denote by $\text{div } u$ the divisibility of $[u] \in G/\text{Tors}(G)$.

Denote by $B(H_3)$ the space of bilinear forms $H_3 \times H_3 \rightarrow \mathbb{Z}$. For $l \in B(H_3)$ denote by $\bar{l} : H_3 \rightarrow H_1$ the adjoint homomorphism uniquely defined by the property $l(x, y) = x \cap_N \bar{l}y$. A pair $(u, l) \in H_2 \times B(H_3)$ is called *symmetric* if

$$l(y, x) = l(x, y) + u \cap_N x \cap_N y \quad \text{for all } x, y \in H_3.$$

The maps $\varkappa_{\#} : E_{\#}^7(N) \rightarrow H_2^{DIFF}$, $\lambda_{\#} : E_{\#}^7(N) \rightarrow B(H_3)$ and $\beta_{u, l, \#}$ of Theorem 1.2 below are defined in §2.2, §2.2 and §2.3 respectively. The map $\varkappa_{\#} : E_{mk}^7(N) \rightarrow H_2$ is well-defined by $\varkappa = \varkappa_{\#} q_{\#}$ because of the additivity of \varkappa (Lemma 2.3 below). We denote this map shortly by \varkappa to avoid double statements of similar properties: a statement involving \varkappa holds for both \varkappa and $\varkappa_{\#}$. If a statement holds for $\varkappa_{\#}$ but not for \varkappa , we write $\varkappa_{\#}$ in the formulation. In this paper there are no statements which hold for \varkappa but not for $\varkappa_{\#}$. Analogous remark holds for λ vs $\lambda_{\#}$ etc.

⁷Suppose that $H_1 = 0$. The forgetful map from $E_{\#}^7(N)$ to the set of PL isotopy classes is injective for $H_1 = 0$ [Bo71, p. 141], [Ha68]. Boéchat and Haeffliger classified PL embeddings $f : N \rightarrow S^7$ up to PL isotopy [BH70, Theorem 1.6]. They also characterized smoothable PL embeddings [BH70, Theorem 2.1]. All this implies the above result. An alternative proof of the injectivity of $\varkappa_{\#}$ is given in [CS11].

Theorem 1.2. *Let N be a closed connected orientable 4-manifold with torsion free H_1 . Then the product*

$$\varkappa_{\#} \times \lambda_{\#} : E_{\#}^7(N) \rightarrow H_2^{DIFF} \times B(H_3)$$

has image consisting of symmetric pairs, and for each $(u, l) \in \text{im}(\varkappa_{\#} \times \lambda_{\#})$ with $d := \text{div } u$, the map

$$\beta_{u,l,\#} : (\varkappa_{\#} \times \lambda_{\#})^{-1}(u, l) \rightarrow \text{coker}(2\rho_d \bar{l}) = \frac{H_1}{2\bar{l}(H_3) + dH_1}$$

is a bijection.

The Seifert bilinear form $\lambda(f) : H_3 \times H_3 \rightarrow \mathbb{Z}$ (defined in §2.2) measures the linking of 3-cycles in N under f . For $N = S^1 \times S^3$ identify $B(H_3)$ with \mathbb{Z} .

Lemma 1.3 (Calculation for λ ; proved in §2.2). *(a) For an embedding $f : S^1 \times S^3 \rightarrow S^7$ we have*

$$\lambda[f] = \text{lk}_{S^7}(f|_{(1,0) \times S^3}, f|_{(-1,0) \times S^3}) \in \mathbb{Z}.$$

(b) $\lambda(\tau(l, b)) = l$.

(c) We have $\lambda(f)(x, y) = \text{lk}_{S^7}(f|_X, f|_Y)$ if classes x and y are represented by disjoint closed oriented 3-submanifolds (or integer 3-cycles) X and Y .

In §1.4 we explain how the invariants appear in our approach to classification.

Remark 1.4. (a) Theorem 1.1 is not an immediate corollary of Theorem 1.2, see Remark 2.16.

(b) For each $u \in H_2^{DIFF}$ the set $\lambda(\varkappa^{-1}(u))$ consists of those $l \in B(H_3)$ for which (u, l) is symmetric.

(c) For fixed u , the set of symmetric pairs (u, l) is in bijection with the group $B_0(H_3)$ of symmetric bilinear forms $H_3 \times H_3 \rightarrow \mathbb{Z}$. Indeed $B_0(H_3)$ acts freely and transitively on this set, for if (u, l) and (u, l') are symmetric pairs, then $l - l'$ is a symmetric form.

(d) Theorem 1.2 has a restatement similar to Theorem 1.1, see Corollary 2.14.b.

1.3 The Knotting Problem

In this subsection we provide a broader context for the results in this paper. The classical Knotting Problem runs as follows: *given an n -manifold P and a number m , describe $E^m(P)$, the set of isotopy classes of embeddings $P \rightarrow S^m$.* For recent surveys see [Sk08, MAH]; whenever possible we refer to these surveys not to original papers.

The Knotting Problem is more accessible for

$$2m \geq 3n + 4,$$

where there are some classical complete readily calculable classifications of embeddings, which are surveyed in [Sk08', §2, §3], [MAH].

The Knotting Problem is much harder for $2m < 3n + 4$. If P is a closed manifold that is not a disjoint union of homology spheres, then until recently no complete readily calculable descriptions of $E^m(P)$ was known. This is in spite of the existence of many interesting approaches including methods of Haefliger-Wu, Browder-Wall and Goodwillie-Weiss [Sk08, §5], [Wa70, GW99, CRS04] (cf. [Sk10, footnote 2]).

For $m \geq n + 3$, $E^m(S^n)$ is a group under embedded connected sum, defined analogously to the definition for $E^7(S^4)$ in §1.1. This group again acts on $E^m(P)$ by embedded connected sum as in §1.1. A simpler version of the knotting problem is to pass to the quotient

$$E_{\#}^m(P) := E^m(P) / E^m(S^n).$$

But even the simpler problem of determining $E_{\#}^m(P)$ is hard: If $2m < 3n + 4$ and P is a closed manifold that is not $[(n-2)/2]$ -connected, then until recently no complete readily calculable description of $E_{\#}^m(P)$ was known.

Recent complete readily calculable classification results for $2m < 3n + 4$ concern

- embeddings of 3- and 4-dimensional manifolds [Sk08', Sk10, CS11],
- embeddings of d -connected n -manifolds for $2m \geq 3n + 3 - d$ [Sk02], and
- embeddings $S^p \times S^q \rightarrow S^m$ [CRS07, CRS12, CFS14, Sk15].

These results are based on three fruitful approaches. One of them involves almost embeddings and the β -invariant of [Sk02, Sk07, Sk14, CRS07, CRS12] (which, though related to, is different from the β -invariant in this paper), another is based on relations between different sets of embeddings [Sk11, Sk15]. However, these and other approaches are not sufficient to classify 4-manifolds N into S^7 , even in the case of $N = S^1 \times S^3$. In this paper we apply the approach which uses Kreck's modified surgery, cf. [Sk08', Sk10, CS11].

There is a map

$$\tau : \pi_q(V_{m-q,p+1}) \rightarrow E^m(S^p \times S^q)$$

which is defined analogously to §1.2. For $m \geq 2p + q + 3$ the sets $E^m(S^p \times S^q)$ and $E_{\#}^m(S^p \times S^q)$ possesses a group structure such that τ and $q_{\#}$ are homomorphisms [Sk15]. For $p \leq q$ and $2m \geq 2p + 3q + 4$ (conjecturally for $2m \geq p + 3q + 4$) $\tau_{\#}$ is an isomorphism. Theorem 1.1 shows that the case of embeddings $S^1 \times S^3 \rightarrow S^7$ is different:

- there are no group structures on $E^7(S^1 \times S^3)$ such that τ is a homomorphism (because by Theorem 1.1 and the fact that $|E^7(S^4)| = 12$, the preimages of τ vary in size);
- there are no group structures on $E_{\#}^7(S^1 \times S^3)$ such that $\tau_{\#}$ is a homomorphism (because by Theorem 1.1, the preimages of $\tau_{\#}$ vary in size).

1.4 An approach to the Knotting Problem

The proofs of our main results are based on the ideas we explain below. These ideas are useful in a wider range of dimensions [Sk08'] and for problems other than classification of embeddings [Kr99]. Except for the notation and Lemma 1.5, the material of this subsection is not formally used in the rest of this paper.

Some notation. Take the standard orientation on \mathbb{R}^m . For an oriented manifold with boundary we use the orientation on the boundary whose completion by ‘the first vector pointing outside’ gives the orientation on the manifold. So an orientation of $S^{m-1} = \partial D^m$ is defined. Fix an orientation on N . Denote by

- $C = C_f$ the closure of the complement in $S^7 \supset \mathbb{R}^7$ to a sufficiently small tubular neighborhood of $f(N)$; the orientation on C is inherited from the orientation of S^7 ;
- $\nu = \nu_f : \partial C \rightarrow N$ the sphere subbundle of the normal vector bundle of f : the total space of ν is identified with ∂C ;

In this paper a *bundle isomorphism* is an oriented vector bundle isomorphism identical on the base, or the restriction to the sphere bundle of such. In this and other notation we sometimes omit the subscript f . We shall also change the subscript ‘ f_k ’ to ‘ k ’.

Lemma 1.5. *For a closed connected 4-manifold N two embeddings $f_0, f_1 : N \rightarrow S^7$ are isotopic if and only if there is an orientation preserving diffeomorphism $C_0 \rightarrow C_1$ whose restriction to the boundary $\partial C_0 \rightarrow \partial C_1$ is a bundle isomorphism.*

Lemma 1.5 is classical (for a proof see, e.g., [Sk10, Lemma 1.3]).

Remark 1.6. We shall not only decide if there is a diffeomorphism $C_0 \rightarrow C_1$ as in Lemma 1.5 but we also prove a general ‘relative diffeomorphism criterion’ for certain 7-manifolds with boundary. This is the Almost Diffeomorphism Theorem 4.5. It generalizes [CS11, Almost Diffeomorphism

Theorem 2.8, Diffeomorphism Theorem 4.7]. It is a new non-trivial analogue of [KS91, Theorem 3.1] and of [Kr99, Theorem 6] for 7-manifolds M with non-empty boundary and infinite $H_4(M)$.

Lemma 1.5 reduces the classification of embeddings to a two-step classification problem for their complements. Firstly, we classify the complements relative to fixed identifications of their boundaries, and secondly we determine the action of the bundle automorphisms on the relative diffeomorphism classes of the complements. This is the starting point of both the classical and modified surgery approaches. As we continue this introduction, we shall not assume the reader is familiar with surgery. Hence we describe the application of modified surgery in non-specialist terms and make parenthetical remarks for specialists.

To decide if there is a diffeomorphism $C_0 \rightarrow C_1$ as in Lemma 1.5 using classical surgery (see [Wa70]), we would first need decide if the complements have the same homotopy type. If they do, then we take homotopy equivalences $h_k : C_k \rightarrow B$, $k = 0, 1$, and apply surgery relative to the boundary to the Poincaré pairs $(B, h_0(\partial C_0))$ and $(B, h_1(\partial C_1))$.

In this paper to determine if there is a diffeomorphism $C_0 \rightarrow C_1$ as in Lemma 1.5 we use modified surgery [Kr99]; cf. [CS11, Remark 2.2] and the text after it. For this we fix for $k = 0, 1$

- the spin structures on C_k which they inherit from S^7 and also
- the *Seifert classes*, i.e. the relative homology classes $A_k[N] \in H_5(C_k, \partial C_k) \cong \mathbb{Z}$, which are the images of the fundamental class of N under homological Alexander duality (defined in §3.1).

(This data on C_k defines a *normal 2-smoothing* of C_k [Kr99, §2], i.e. a normal 3-equivalence $C_k \rightarrow BSpin \times \mathbb{C}P^\infty$. We use a particular case of the modified surgery approach which corresponds to spin surgery over the *homotopy 2-type* of the complement.)

The modified surgery approach to the embedding problem requires that we find a bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ preserving both the spin structures and the homology classes $\partial A_k[N] \in H_3(\partial C_k)$. We prove that *there is always a bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ preserving spin structure* (Lemma 3.6, cf. [CS11, Lemma 2.4]). The first obstruction we encounter to the existence of a diffeomorphism as in Lemma 1.5 is the difference $\varphi_* \partial A_0[N] - \partial A_1[N] \in H_3(\partial C_1)$. The analysis of this obstruction leads to the definition of \varkappa -invariant (§2.2)⁸

$$\varkappa : E^7(N) \rightarrow H_2.$$

Assume further that $\varkappa(f_0) = \varkappa(f_1)$. We prove that $\varphi_* \partial A_0[N] = \partial A_1[N]$ for each bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ (Lemma 3.5.a for $q = 4$, [CS11, Agreement Lemma 2.5]). We then identify the spin boundaries of $(C_0, A_0[N])$ and $(C_1, A_1[N])$ via a bundle isomorphism φ which preserves the spin structures. For any such identification *there is a spin bordism (W, z) between $(C_0, A_0[N])$ and $(C_1, A_1[N])$ relative to the boundaries* (because the complete obstruction to the existence of such a bordism assumes values in $\Omega_7^{Spin}(\mathbb{C}P^\infty) = 0$ [KS91, Lemma 6.1]). It remains to determine whether we can replace the bordism (W, z) by an h -cobordism. This problem is addressed in [Kr99, Theorem 3], where a complete algebraic obstruction is defined. Analysis of the obstruction for the bordism (W, z) to have the homology of an h -cobordism ‘outside $H_4(W)$ ’ leads to the definition of λ -invariant in (§2.2)

$$\lambda : E^7(N) \rightarrow B(H_3).$$

Assume further that $\lambda(f_0) = \lambda(f_1)$. From the surgery point of view, the β -invariant (see definition in §2.3) arises as the obstruction for the bordism (W, z) to have the homology of an h -cobordism ‘in the summand of $H_4(W)$ coming from $H_4(\partial W)$ ’ (i.e. in the *singular* part of the intersection form on $H_4(W)$). This invariant assumes values in a quotient of H_1 defined by $\varkappa(f_0)$ and $\lambda(f_0)$.

Assume further that $\beta(f_0) = \beta(f_1)$. We may now assume that the bordism (W, z) ‘has the homology of an h -cobordism’ away from the *unimodular* part of $H_4(W)$. Extending arguments

⁸Note that \varkappa -invariant can alternatively be defined using the intersection in the homology of the complements C_0, C_1 . However, that definition corresponds to the classical surgery not modified surgery approach (a similar remark can be made for the λ -invariant described below).

from [CS11], we prove that we can modify f_0 by connected sum with a knot $g: S^4 \rightarrow S^7$ so that for some corresponding

- spin bundle isomorphism $\varphi' : \partial C_{f_0 \# g} \rightarrow \partial C_1$,
- identification of the spin boundaries of the pairs $(C_{f_0 \# g}, A_{f_0 \# g}[N])$ and $(C_1, A_1[N])$,
- spin null bordism (W', z') between the above pairs, relative to the boundaries,

the pair (W', z') is bordant to an h -cobordism. Then by the h -cobordism theorem [Mi65] and Lemma 1.5, $f_0 \# g$ and f_1 are isotopic.

The above discussion outlines the proof that the κ -, λ - and β -invariants combine to give a complete systems of invariants for embeddings modulo knots. This is stated in the MK Isotopy Classification Theorem 2.8 and the behaviour of these invariants under connected sum with knots is described in the Additivity Lemmas 2.3, 2.9.

Plan of the paper. We introduce further notation in §2.1 and §3.1. In §2 we present the important constructions and lemmas used in the proof of our main results. The lemmas from §2 are proven in §3 and §4. The subsection titles in §3 indicate the most important lemmas proven in that subsection. A reader who wants to check a particular lemma from §2 does not need to read all of §3 and §4.

Acknowledgments. We would like to acknowledge S. Avvakumov, M. Kreck, S. Melikhov and D. Tonkonog for useful discussions. We would like to thank the Hausdorff Institute for Mathematics and the University of Bonn for their hospitality and support during the early stages of this project.

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2 Definitions of the invariants and proofs modulo lemmas

2.1 Main notation

Recall that some notation was introduced in §1.2 and §1.4.

Some identifications.

Identify $\pi_n(S^n)$ and \mathbb{Z} by the degree isomorphism.

Identify S^2 and $\mathbb{C}P^1$. Represent $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. The Hopf map $\eta : S^3 \rightarrow S^2$ is defined by $\eta(z_1, z_2) = [z_1 : z_2]$. Identify $\pi_3(S^2)$ and \mathbb{Z} by the Hopf isomorphism (that sends the homotopy class of η to 1).

Identify \mathbb{R}^4 and the algebra \mathbb{H} of the quaternions. Identify $\pi_3(V_{4,2})$ and $\pi_3(S^3) \oplus \pi_3(S^2) = \mathbb{Z}^2$ by the standard isomorphism, which is defined using the projection $V_{4,2} \rightarrow S^3$ given by $(x, y) \rightarrow x$ and the section $S^3 \rightarrow V_{4,2}$ given by $x \mapsto (x, xi)$. Identify $\pi_3(SO_3)$ and $\pi_3(S^2) = \mathbb{Z}$ by the map induced by the action of SO_3 on S^2 .

General notation.

Denote by

- N a closed connected orientable 4-manifold *with torsion free* H_1 ;
- $f, f_0, f_1 : N \rightarrow S^7$ embeddings;
- \equiv_n a congruence modulo n ;
- pr_k the projection of a Cartesian product onto the k -th factor;
- $\text{id } X$ the identity map of the set X ;
- $1_m := (1, 0, \dots, 0) \in S^m$;
- $\text{Cl } X$ the closure of a subset X in the ambient space, which is clear from the context;
- $N_0 := \text{Cl}(N - B^4)$, where B^4 is an embedded closed 4-ball in N .

For each $q \leq m$ identify the space \mathbb{R}^q with the subspace of \mathbb{R}^m given by the equations $x_{q+1} = x_{q+2} = \dots = x_m = 0$. Analogously identify D^q, S^{q-1} with the corresponding subspaces of D^m, S^{m-1} .

Define $\mathbb{R}_+^m, \mathbb{R}_-^m \subset \mathbb{R}^m$ and $D_+^m, D_-^m \subset S^m$ by the equations $x_1 \geq 0$ and $x_1 \leq 0$, respectively. Then

$$S^m = D_+^m \bigcup_{\partial D_+^m = \partial D_-^m} D_-^m \quad \text{and} \quad \partial D_+^m = \partial D_-^m = D_+^m \cap D_-^m = 0 \times S^{m-1} \neq S^{m-1}.$$

We denote the union of oriented manifolds in the same way as set-theoretic union. So both formulas $S^4 = D_+^4 \cup (-D_-^4)$ and $S^4 = D_+^4 \cup D_-^4$ are correct, the sign “ \cup ” means union of oriented manifolds in the first formula and union of manifolds in the second one.

Homological notation.

Denote by $[\cdot]$ the homology class or equivalence class in a quotient group.

We denote the maps induced in homology by the same letters as the inducing maps. Thus if $f : X \rightarrow Y$ is a map of spaces, $f : H_*(X) \rightarrow H_*(Y)$ denotes the induced map on homology.

Homomorphisms between homology groups with \mathbb{Z}_d -coefficients are denoted in the same way as those for \mathbb{Z} -coefficients. So the coefficients are to be understood from the context. When this could lead to confusion, we specify coefficients by indicating the domain and the range of the homomorphism, e.g. $i : H_3(C_0; \mathbb{Z}_d) \rightarrow H_3(M_\varphi; \mathbb{Z}_d)$.

We denote by $i_{A,X}, j_{A,X}, \partial_{A,X}$ or shortly by i_A, j_A, ∂_A or shortly by i, j, ∂ , the homomorphisms from the exact sequence of the pair (X, A) . If $A = C_k$ or $A = C_f$, then we shorten the subscript C_k or C_f to just k or just f , respectively. Denote by $\text{ex}: H_q(X, A) \rightarrow H_q(X - B, A - B)$ the excision isomorphism, where B is a subset of A .

For a compact p -manifold P denote $H_q(P, \partial) := H_q(P, \partial P)$.

Let P and Q be compact oriented p - and q -manifolds. Denote by

$$\text{PD}: H^n(P) \rightarrow H_{p-n}(P, \partial) \quad \text{and} \quad \text{PD}: H^n(P, \partial) \rightarrow H_{p-n}(P),$$

the Poincaré duality isomorphisms. We sometimes identify homology and cohomology groups by Poincaré duality. We choose to work mostly with homology classes, since this has technical advantages for our arguments, see [CS11, Remark 2.3].

For a map $\xi: P \rightarrow Q$ denote the ‘preimage’ homomorphism by

$$\xi^! := \text{PD} \circ \xi \circ \text{PD}^{-1}: H_n(Q, \partial) \rightarrow H_{p-q+n}(P, \partial),$$

where ξ is the homomorphism induced in cohomology.

We now consider the intersection of and linking numbers of singular (or more general) chains.

For set-theoretic intersection we write $X \cap Y$. (This notation is also used for restriction, see §3.1.) For the algebraic intersection of chains or integer cycles or oriented manifolds in an ambient manifold M we write $X \cap_M Y$. Recall that $X \cap_M Y = (-1)^{\text{codim } X \text{ codim } Y} Y \cap_M X$, and that if X, Y are cycles, then $X \cap_M Y$ depends only on the homology classes represented by X and Y .

Let A and B be integer a - and b -cycles in \mathbb{R}^m having disjoint supports with $a + b + 1 = m$. Define the *linking number* of A and B by $\text{lk}(A, B) := A \cap_{\mathbb{R}^m} \beta$, where β is a $(b+1)$ -cycle in \mathbb{R}^m with $\partial\beta = B$. It is easy to check that $\text{lk}(A, B) = \alpha \cap_{\mathbb{R}^m} B$, where α is an $(a+1)$ -cycle in \mathbb{R}^m with $\partial\alpha = A$. Recall that $\text{lk}(A, B) = (-1)^{(m-a)(m-b)} \text{lk}(B, A)$.

2.2 Definitions of the κ - and λ -invariants

Definition of a weakly unlinked section of an embedding f . Let $\zeta: N_0 \rightarrow \nu^{-1}N_0$ be a section of $\nu^{-1}N_0 \rightarrow N_0$, the restriction of ν to N_0 . The composition $N_0 \xrightarrow{\zeta} \nu^{-1}N_0 \xrightarrow{\subseteq} \partial C$ of ζ and the inclusion is called a *weakly unlinked section* provided $i_C j_{\partial C}^{-1} \text{ex}^{-1} \zeta = 0 \in H_4(C)$. Here the homomorphism $i_C j_{\partial C}^{-1} \text{ex}^{-1} \zeta$ can be obtained by inverting the isomorphisms $j_{\partial C}$ and ex in the following diagram:

$$\mathbb{Z} \cong H_4(N_0, \partial) \xrightarrow{\zeta} H_4(\nu^{-1}N_0, \partial) \xleftarrow{\text{ex}} H_4(\partial C, \nu^{-1}B^4) \xleftarrow{j_{\partial C}} H_4(\partial C) \xrightarrow{i_C} H_4(C).$$

We remark that

- a section $\zeta: N_0 \rightarrow \nu^{-1}N_0$ exists because the Euler class of ν is zero, vector bundle associated to ν is 3-dimensional and N_0 retracts to a 3-polyhedron;
- any section $\zeta: N_0 \rightarrow \nu^{-1}N_0$ is weakly unlinked for $N = S^1 \times S^3$ because there is an isomorphism $H_4(S^7 - f(S^1 \times S^3)) \cong H_2(S^1 \times S^3) = 0$. Cf. Lemma 3.3.a.

Lemma 2.1. *A weakly unlinked section exists and is unique up to vertical homotopy over the 2-skeleton of any triangulation of N .*

Proof. This holds by [BH70, Proposition 1.3] because by [Sk10, Remark 2.4 and footnote 14] our definition of a weakly unlinked section is equivalent to the original definition [BH70]. Cf. proof of Lemma 3.3.b and [Sk08', the Unlinked Section Lemma (a)]. \square

Definition of the Boéchat-Haefliger invariant $\kappa: E^7(N) \rightarrow H_2$. Represent a class $x \in H_2$ by a closed oriented 2-submanifold (or integer 2-cycle) $X \subset N_0$. Take a weakly unlinked section

$\xi : N_0 \rightarrow \partial C$. By Poincaré duality $\cap : H_2 \times H_2 \rightarrow \mathbb{Z}$ is unimodular. Since H_2 is torsion-free, it follows that $\varkappa(f)$ is uniquely defined by the equation

$$\varkappa(f) \cap_N x = \text{lk}_{S^7}(fN, \xi X),$$

for all $x \in H_2$. This is well-defined; i.e. is independent of the choice of ξ , by Lemma 3.2. \varkappa' . This definition is equivalent to those of [BH70, Sk10, CS11] by Lemma 3.2. \varkappa' ,e.⁹ Clearly, the map $\varkappa : E^7(N) \rightarrow H_2$ is well-defined by $\varkappa([f]) := \varkappa(f)$.

Definition of the Seifert form $\lambda : E^7(N) \rightarrow B(H_3)$. Represent classes $x, y \in H_3$ by closed oriented 3-submanifolds (or integer 3-cycles) $X, Y \subset N_0$. Take a weakly unlinked section $\xi : N_0 \rightarrow \partial C$. Define

$$\lambda(f)(x, y) := \text{lk}_{S^7}(fX, \xi Y) \in \mathbb{Z}.$$

This is well-defined; i.e. is independent of the choice of ξ , by Lemma 3.2. λ' . Clearly, the pairing $\lambda(f) : H_3 \times H_3 \rightarrow \mathbb{Z}$ is indeed a bilinear form. Clearly, the map $\lambda : E^7(N) \rightarrow B(H_3)$ is well-defined by $\lambda([f]) := \lambda(f)$. Cf. [Sa99, To10].

Proof of Lemma 1.3. Part (c) follows because ξY and fY are homologous in $S^7 - fX$. Part (a) follows by (c). Part (b) follows by (a). \square

We give equivalent definitions of \varkappa and λ in Lemma 3.2.

Lemma 2.2 (\varkappa -symmetry; proved in §3.2). *We have $\lambda(f)(y, x) = \lambda(f)(x, y) - \varkappa(f) \cap_N x \cap_N y$.*

Cf. [Sa99, Lemma 2.2], [To10, Theorem 1.5(2) and Lemmas 1.6 and 2.10].

Lemma 2.3 (Additivity of λ and \varkappa). *For every pair of embeddings $g : S^4 \rightarrow S^7$ and $f : N \rightarrow S^7$*

$$\varkappa(f \# g) = \varkappa(f) \quad \text{and} \quad \lambda(f \# g) = \lambda(f).$$

Proof. We may assume that $g(S^4) \cap C_f = \emptyset$ and $\nu_f = \nu_{f \# g}$ over N_0 . Then additivity for λ and \varkappa follows because a weakly unlinked section for f is also a weakly unlinked section for $f \# g$. \square

2.3 Definition of the β -invariant and the map $\beta_{u,l,\#}$

Definitions of a meridian, the manifold $M = M_\varphi$ and of a joint Seifert class. Take a small oriented disk $D_f^3 \subset \mathbb{R}^7$ whose intersection with $f(N)$ consists of exactly one point of sign +1 and such that $\partial D_f^3 \subset \partial C_f$. Define *the meridian of f* by

$$S_f^2 := [\partial D_f^3] \in H_2(C_f).$$

By Alexander duality S_f^2 is a generator of $H_2(C_f)$.

For a bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ define the closed oriented 7-manifold

$$M = M_\varphi := C_0 \cup_\varphi (-C_1).$$

A *joint Seifert class* for a bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ is a class

$$Y \in H_5(M_\varphi) \quad \text{such that} \quad Y \cap_{M_\varphi} i_{\partial C_0, M_\varphi} S_{f_0}^2 = 1.$$

We shall omit the phrase ‘for a bundle isomorphism φ ’ if its choice is clear from the context.

We remark that the property of Y identified in Lemma 3.13.a below provides an equivalent definition of a joint Seifert class which explains the name and which was used in [Sk10, CS11], together with the name ‘joint homology Seifert surface’.

⁹Those definitions do not require the assumption that H_1 is torsion free. In those papers the invariant was denoted by w_f [BH70], or $BH(f)$ [Sk10], or $\aleph(f)$ [CS11], instead of $\varkappa(f)$.

Lemma 2.4 (proved in §3.4). *If $\varkappa(f_0) = \varkappa(f_1)$ and $\varphi: \partial C_0 \rightarrow \partial C_1$ is a bundle isomorphism, then there is a joint Seifert class $Y \in H_5(M_\varphi)$.*

Definition of a π -isomorphism. We call a bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$ a π -isomorphism if M_φ is parallelizable.

Lemma 2.5 (proved in §3.6). *If $\varkappa(f_0) = \varkappa(f_1)$ and $\lambda(f_0) = \lambda(f_1)$, then there is a π -isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$. A π -isomorphism is unique (up to vertical homotopy through linear isomorphisms) over N_0 .*

Definitions of $K_{u,l}$ and $\beta(f_0, f_1)$. For $u \in H_2$ and $l \in B(H_3)$ recall that $\bar{l}: H_3 \rightarrow H_1$ is the adjoint of l and that $\rho_d: H_1 \rightarrow H_1(N; \mathbb{Z}_d)$ is reduction modulo d . Define

$$K_{u,l} := \text{coker}(2\rho_{\text{div } u} \bar{l}).$$

Assume that $\varkappa(f_0) = \varkappa(f_1)$ and that $\lambda(f_0) = \lambda(f_1)$. Denote $d := \text{div}(\varkappa(f_0))$. By Lemmas 2.4 and 2.5 there is a joint Seifert class $Y \in H_5(M_\varphi)$ and a π -isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$. Define

$$\beta(f_0, f_1) := [(i_{\partial C_0, M_\varphi} \nu_0^\dagger)^{-1} \rho_d Y^2] \in K_{\varkappa(f_0), \lambda(f_0)}$$

using the composition

$$H_1(N; \mathbb{Z}_d) \xrightarrow{\nu_0^\dagger} H_3(\partial C_0; \mathbb{Z}_d) \xrightarrow{i_{\partial C_0, M_\varphi}} H_3(M_\varphi; \mathbb{Z}_d).$$

Lemma 2.6 (proved in §3.7). *The class $\beta(f_0, f_1)$ is well-defined; i.e.*

- for each joint Seifert class Y and π -isomorphism φ there is a unique element $b_{\varphi, Y} \in H_1(N; \mathbb{Z}_d)$ such that $i_{\partial C_0, M_\varphi} \nu_0^\dagger b_{\varphi, Y} = \rho_d Y^2 \in H_3(M_\varphi; \mathbb{Z}_d)$,
- $[b_{\varphi, Y}] \in K_{\varkappa(f_0), \lambda(f_0)}$ is independent of the choice of joint Seifert class Y and π -isomorphism φ .

Lemma 2.7 (Calculation of β ; proved in §3.8). (a) $\beta(\tau(0, 0), \tau(0, b)) = b[S^1 \times 1_3] \in H_1(S^1 \times S^3)$.
(b) $\beta(\tau(l, b'), \tau(l, b)) = \rho_{2l}(b - b')[S^1 \times 1_3] \in H_1(S^1 \times S^3; \mathbb{Z}_{2l})$ (cf. Remark before Lemma 3.15).

Theorem 2.8 (Isotopy Classification Modulo Knots; proved in §4.2). *If we have $\lambda(f_0) = \lambda(f_1)$, $\varkappa(f_0) = \varkappa(f_1)$ and $\beta(f_0, f_1) = 0$, then there is an embedding $g: S^4 \rightarrow S^7$ such that f_0 is isotopic to $f_1 \# g$.*

Lemma 2.9 (Additivity of β ; proved in §3.7). *For every pair of embeddings $g: S^4 \rightarrow S^7$ and $f: N \rightarrow S^7$ we have $\beta(f \# g, f) = 0$.*

Lemma 2.10 (Transitivity of β ; proved in §3.7). *For every triple of embeddings $f_0, f_1, f_2: N \rightarrow S^7$ having the same values of \varkappa - and λ -invariants we have $\beta(f_2, f_0) = \beta(f_2, f_1) + \beta(f_1, f_0)$.*

Definitions of the maps $\beta, \beta_\#$ and $\beta_{u,l,\#}$. Let us define maps

$$\beta: [(\varkappa \times \lambda)^{-1}(u, l)]^2 \rightarrow K_{u,l}, \quad \beta_\#: [(\varkappa_\# \times \lambda_\#)^{-1}(u, l)]^2 \rightarrow K_{u,l}, \quad \beta_{u,l,\#}: (\varkappa_\# \times \lambda_\#)^{-1}(u, l) \rightarrow K_{u,l}.$$

Clearly, the map β is well-defined by $\beta([f], [g]) := \beta(f, g)$.

The map $\beta_\#$ is well-defined by $\beta = \beta_\#(q_\# \times q_\#)$, according to the additivity and the transitivity of β (Lemmas 2.9 and 2.10).

Take an embedding $f': N \rightarrow S^7$ representing an isotopy class in $(\varkappa_\# \times \lambda_\#)^{-1}(u, l)$. Let $\beta_{u,l,\#}[f] := \beta(f, f')$. The map $\beta_{u,l,\#}$ depends on f' but we do not indicate this in the notation.

2.4 Parametric connected sum and parametric additivity

In our proof of the realization of the invariants we extensively use the *parametric connected sum operation* defined below. We first recall, with minor modifications, some definitions and results of [Sk07, §2], [Sk15, §2.1], [MAP].

Definition of a standardized map. The base point $*$ of $V_{4,2}$ is the standard inclusion $\mathbb{R}^2 \rightarrow \mathbb{R}^4$. Take the embedding $\tau_0 : S^1 \times S^3 \rightarrow S^7$ for the constant map $\alpha_0 : S^3 \rightarrow V_{4,2}$ (as defined in §1.2). Clearly, $\tau_0(S^1 \times D_{\pm}^3) \subset D_{\pm}^7$. For an embedding $s : S^1 \times D_-^3 \rightarrow N$, a map $h : N \rightarrow S^7$ is called *s-standardized* if

$$h(N - \text{im } s) \subset \text{Int } D_+^7 \quad \text{and} \quad h \circ s = \tau_0|_{S^1 \times D_-^3}.$$

Lemma 2.11. *For each embedding $s : S^1 \times D_-^3 \rightarrow N$, f is isotopic to an s -standardized embedding $\tilde{f} : N \rightarrow S^7$.*

This is a smooth version of [Sk07, Standardization Lemma] which is proved analogously, cf. [Sk15, Standardization Lemma 2.1.a]. We present the proof in §3.3 for the reader's convenience.

Definition of parametric connected sum $[f] +_s \tau(l, b)$. Take a map $\alpha : S^3 \rightarrow V_{4,2}$ representing an element $(l, b) \in \mathbb{Z}^2 = \pi_3(V_{4,2})$ such that $\alpha(D_-^3) = *$. Then the embedding τ_α is i -standardized for the inclusion $i : S^1 \times D_-^3 \rightarrow S^1 \times S^3$. Take an embedding $s : S^1 \times D_-^3 \rightarrow N$ and an s -standardized embedding $\tilde{f} : N \rightarrow S^7$ isotopic to f . Let $R : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the symmetry of \mathbb{R}^m with respect to the hyperplane given by equations $x_1 = x_2 = 0$, i.e., R is defined by $R(x_1, x_2, x_3, \dots, x_m) := (-x_1, -x_2, x_3, \dots, x_m)$. Define the embedding

$$h : N \rightarrow S^7 \quad \text{by} \quad h(a) := \begin{cases} \tilde{f}(a) & a \notin \text{im } s, \\ R\tau_\alpha(x, Ry) & a = s(x, y). \end{cases}$$

The two formulas agree on $\partial \text{im } s$ because $\tau_0(x, y) = R\tau_0(x, Ry)$. Clearly, h is a smooth embedding, i.e. it is injective, differentiable and has non-degenerate derivative.

Let $[f] +_s \tau(l, b) \subset E^7(N)$ be the set of isotopy classes of the embeddings h , for all choices of \tilde{f} and α as above. (In fact, $[h]$ clearly does not depend on the choice of α , for fixed l, b, \tilde{f}, s . Still, $[h]$ may depend on \tilde{f} , i.e. $+_s$ is not defined at the level of isotopy for embeddings of 4-manifolds into S^7 , as opposed to other situations [Sk07, Sk15, MAP]. Cf. Corollary 2.14.c,d,e.)

Remark 2.12. Note that $[f] \in [f] +_s \tau(0, 0)$ for every s . Also $\tau(l+l', b+b') \in \tau(l, b) +_{[S^1 \times 1_3]} \tau(l', b')$, and in particular, $\tau(l, b') \in \tau(l, b) +_{[S^1 \times 1_3]} \tau(0, b' - b)$.

Lemma 2.13 (Parametric additivity; proved in §3.3). *For any embedding $s : S^1 \times D_-^3 \rightarrow N$ let $[s] := [s|_{S^1 \times 0_3}] \in H_1$. Then for any embedding $h \in f +_s \tau(l, b)$ and $x, y \in H_3$ we have*

$$\kappa(h) = \kappa(f), \quad \lambda(h)(x, y) = \lambda(f)(x, y) + l([s] \cap_N x)([s] \cap_N y) \quad \text{and,}$$

$$\text{for } l = 0, \quad \beta(f, h) = b[s] \in K_{\kappa(f), \lambda(f)}.$$

Cf. The remark before Lemma 3.15.

Corollary 2.14. (a) *For every $u \in H_2^{DIFF}$ every isotopy class in $\kappa^{-1}(u)$ can be obtained from a single isotopy class in $\kappa^{-1}(u)$ by a finite sequence of parametric connected sum operations.*

(b) *There is a surjection*

$$\tau_{\#} : H_1 \times H_2^{DIFF} \times B_0(H_3) \rightarrow E_{\#}^7(N) \quad \text{such that}$$

$$\tau_{\#}(b, u, l) = \tau_{\#}(b', u', l') \quad \Leftrightarrow \quad u = u', \quad l = l' \quad \text{and} \quad b - b' \in K_{u, l + \lambda_{\#} \tau_{\#}(0, u, 0)}.$$

- (c) The set $q_{\#}([f] +_s \tau(l, b))$ is independent of s for fixed l, b, \tilde{f} and $[s]_{S^1 \times 0_3} \in H_1$ (cf. [Sk14]).
- (d) Both sets $[f] +_s \tau(l, b)$ and $q_{\#}([f] +_s \tau(l, b))$ may consist of more than one element, i.e. $q_{\#}[h]$ from the definition of $[f] +_s \tau(l, b)$ can depend on the choice of \tilde{f} , even for $N = S^1 \times S^3$.
- (e) The set $q_{\#}(\tau(0, b') +_i \tau(0, b))$ consists of one element, i.e. $q_{\#}[h]$ from the definition of $\tau(0, b') +_i \tau(0, b)$ does not depend on the choice of \tilde{f} .

Proof. Parts (a,c) follow from Theorem 1.2 and the parametric additivity (Lemma 2.13). Part (b) follows from (a,c) and Remark 1.4.c. Part (d) follows from Remark 2.12 and the parametric additivity (Lemma 2.13). Part (e) follows from Theorem 1.1 and the parametric additivity (Lemma 2.13). \square

Remark 2.15. (a) In Corollary 2.14.b it would be interesting to canonically construct at least part of the map $\tau_{\#}$, and to give an algebraic (possibly non-canonical) construction of an u -symmetric form instead of $\lambda\tau_{\#}(0, u, 0)$.

(b) Using parametric connected sum one can define a map $E^7(S^1 \times S^3)^2 \rightarrow 2^{E^7(S^1 \times S^3)}$, and the same statement holds with E^7 replaced by $E_{\#}^7$, cf. [Sk15, §2.1], [MAP]. Corollary 2.14.de means that this map

- is not single-valued for either E^7 or $E_{\#}^7$, this is unlike the situation in other dimensions [Sk15, §2.1], [MAP],

- is single-valued on $\lambda_{\#}^{-1}(0) \subset E_{\#}^7(S^1 \times S^3)$; then it defines a group structure on $\lambda_{\#}^{-1}(0)$ (an unpublished direct proof was sketched by S. Avvakumov).

Cf. [CSII, CSIII] for smooth and PL analogues.

2.5 Proof of Theorems 1.1 and 1.2 assuming lemmas

Proof of Theorem 1.1: the surjectivity of $\tau_{\#}$. Take any embedding $f : S^1 \times S^3 \rightarrow S^7$. Identify $B(H_3(S^1 \times S^3))$ with \mathbb{Z} . Denote $l := \lambda(f) \in \mathbb{Z}$. Take a representative $\alpha : S^3 \rightarrow V_{4,2}$ of $(l, 0) \in \pi_3(V_{4,2})$. Then $[\tau_{\alpha}] = \tau(l, 0)$. By the calculation of λ (Lemma 1.3.b) we have $\lambda(\tau_{\alpha}) = l$.

We have $\text{im } \lambda(f) = l\mathbb{Z}[S^1 \times 1_3]$ and $\varkappa(f) = \varkappa(\tau_{\alpha}) = 0$. Hence $K_{\varkappa(f), \lambda(f)} \cong H_1(S^1 \times S^3; \mathbb{Z}_{2l}) \cong \mathbb{Z}_{2l}$ and the class $\beta(f, \tau_{\alpha}) \in K_{\varkappa(f), \lambda(f)}$ is defined. Take an integer b such that $\beta(f, \tau_{\alpha}) = -\rho_{2l}b[S^1 \times 1_3]$. By the transitivity of β (Lemma 2.10) and the calculation of β (Lemma 2.7.b)

$$\beta([f], \tau(l, b)) = \beta(f, \tau_{\alpha}) + \beta(\tau(l, 0), \tau(l, b)) = \rho_{2l}(-b + b)[S^1 \times 1_3] = 0.$$

Hence by the MK Isotopy Classification Theorem 2.8 $q_{\#}[f] = \tau_{\#}(l, b)$. \square

Proof of Theorem 1.1: description of preimages of $\tau_{\#}$. Denote $\tau := \tau_{\#}(l, b)$ and $\tau' = \tau_{\#}(l', b')$. By the calculation of λ (Lemma 1.3.b) we have $\lambda(\tau) = l$ and $\lambda(\tau') = l'$. So for $l = l'$ by the calculation of β (Lemma 2.7.b) we have $\beta(\tau', \tau) = \rho_{2l}(b - b')[S^1 \times 1_3]$. Hence by the MK Isotopy Classification Theorem 2.8

$$\tau = \tau' \iff l = l' \text{ and } \beta(\tau', \tau) = 0 \iff l = l' \text{ and } b \equiv b' \pmod{2l}.$$

\square

Remark 2.16. Theorem 1.1 also follows from the construction of the map τ , the calculation of λ and β (Lemmas 1.3.b and 2.7.b), together with a version of Theorem 1.2 stating that $\beta_{u,l,\#}$ is a 1–1 map for each representative f' of an isotopy class in $(\varkappa_{\#} \times \lambda_{\#})^{-1}(u, l)$ (such a version is essentially proved in the proof of Theorem 1.2); we take $f' = \tau(0, l)$.

Proof of Theorem 1.2. The map $\beta_{u,l,\#}$ is surjective by the parametric additivity (Lemma 2.13) and is injective by the MK Isotopy Classification Theorem 2.8.

By Lemma 3.2.2, our definition of \varkappa is equivalent to that of [BH70], cf. [Sk10, CS11]. Hence by [BH70] $\text{im } \varkappa_{\#} = \text{im } \varkappa = H_2^{DIFF}$. So it remains to prove that for each $u \in \text{im } \varkappa$

$$\lambda(\varkappa^{-1}(u)) = \{l \in B(H_3) : l(y, x) = l(x, y) + u \cap_N x \cap_N y \text{ for each } x, y \in H_3\}.$$

By the \varkappa -symmetry (Lemma 2.2) and Remark 1.4.c this is implied by the following claim.

Claim. *For each embedding $f : N \rightarrow S^7$ and each symmetric bilinear form $m : H_3 \times H_3 \rightarrow \mathbb{Z}$ there is an embedding $g = g(f, m) : N \rightarrow S^7$ such that*

$$\varkappa(g) = \varkappa(f) \quad \text{and} \quad \lambda(g) = \lambda(f) + m.$$

Proof of claim. We can set $g(f, m_1 + m_2) := g(g(f, m_1), m_2)$. Thus it suffices to construct $g(f, m)$ only for basic forms

$$m_p(x, y) = (p \cap_N x)(p \cap_N y) \quad \text{and} \quad m_{p,q}(x, y) = (p \cap_N x)(q \cap_N y) + (p \cap_N y)(q \cap_N x), \quad \text{where } p, q \in H_1.$$

Take embeddings $s, u, v : S^1 \times D_-^3 \rightarrow N$ whose restrictions to $S^1 \times 0$ represent elements $p, q, p+q \in H_1$, respectively. By the parametric additivity (Lemma 2.13) we can take as $g(f, m_p)$ and $g(f, m_{p,q})$ any elements of

$$[f] +_s \tau(1, 0) \quad \text{and} \quad (([f] +_v \tau(1, 0)) +_s \tau(-1, 0)) +_u \tau(-1, 0), \quad \text{respectively,}$$

where the latter set is the set $h_1 +_u \tau(-1, 0)$ for some $h_1 \in h_2 +_s \tau(-1, 0)$ and for some $h_2 \in [f] +_v \tau(1, 0)$. \square

2.6 Aside on regular homotopy and the Compression problem

Proposition 2.17 (Regular homotopy classification). *If $f_0, f_1 : N \rightarrow S^7$ are embeddings and $(\lambda(f_0) - \lambda(f_1))(x, x) \equiv 0 \pmod{2}$ for all $x \in H_3$, then f_0 and f_1 are regular homotopic.*

Define the map $W : E^7(N) \rightarrow H_1(N; \mathbb{Z}_2)$ by $\rho_2 \lambda(f)(x, x) = W(f) \cap_N x$ for all $x \in H_3(N; \mathbb{Z}_2)$. By Proposition 2.17 W induces an injection on the set of regular homotopy classes of embeddings. By Theorem 1.2 $\text{im } W$ consists of those $y \in H_1(N; \mathbb{Z}_2)$ for which there is $u \in H_2^{DIFF}$ and a u -symmetric bilinear form $l \in B(H_3)$ such that $\rho_2 l(x, x) = y \cap_N x$ for each $x \in H_3(N; \mathbb{Z}_2)$. It would be interesting to obtain a more direct description of $\text{im } W$.

Definition of the Whitney invariant W'_0 . (See [Sk10', §1].) The *singular set* of a smooth map $H : X \rightarrow Y$ between manifolds is $S(H) := \{x \in X : d_x H \text{ is degenerate}\}$.

Let $f_0, f_1 : P \rightarrow S^7$ be immersions of a compact 4-manifold P . Take a general position homotopy $H : P \times I \rightarrow S^7 \times I$ between f_0 and f_1 . By general position, $\text{Cl } S(H)$ is a closed 1-submanifold. Define $W'_0(f_0, f_1) := [\text{Cl } S(H)] \in H_1(P, \partial; \mathbb{Z}_2)$.

(It is well-known that $W'_0(f_0, f_1)$ is indeed independent of H for fixed f_0 and f_1 .)

Lemma 2.18. *Let $f_0, f_1 : P \rightarrow \mathbb{R}^7$ be embeddings of a compact oriented 4-manifold P and $X \subset P$ a closed oriented connected 3-submanifold. Take the normal vector field of X in P defined by the orientations of X and P . Let X' be the shift of X along this vector field. Then*

$$W'_0(f_0, f_1) \cap_P \rho_2[X] = \rho_2[\text{lk}_{\mathbb{R}^7}(f_0 X, f_0 X') - \text{lk}_{\mathbb{R}^7}(f_1 X, f_1 X')].$$

Proof. It suffices to prove this equality for $P = X \times I$, $X = X \times 0$ and $X' = X \times 1$. By the strong Whitney Isotopy Theorem [Sk08, Theorem 2.2.b] $f_0|_X$ and $f_1|_X$ are isotopic. Since both sides of the required equality do not change under isotopy of $\text{id } \mathbb{R}^7$, we may assume that $f_0 = f_1$ on X . Take a general position homotopy $H : X \times I \times I \rightarrow \mathbb{R}^7 \times I$ between f_0 and f_1 that is fixed on X . The homotopy H gives a homotopy $G : X \times I \rightarrow \mathbb{R}^7$ of a normal vector field on $f_0(X) \subset \mathbb{R}^7$ (through normal vector fields which are not assumed to be non-zero). Since $H|_{X \times t}$

is an embedding, for each $x \in X$ and $t \in I$ the differential $d_{(x,t)}H$ is degenerate if and only if $G(x, t) = 0$. By general position, $G^{-1}(0) = H(X' \times I) \cap f_0(X)$ is a finite number of points. Then

$$W'_0(f_0, f_1) \cap_P \rho_2[X] = \rho_2|H(X' \times I) \cap f_0(X)| = \rho_2[\text{lk}_{\mathbb{R}^7}(f_0X, f_0X') - \text{lk}_{\mathbb{R}^7}(f_1X, f_1X')].$$

□

Proof of Proposition 2.17. The following statements are equivalent:

- (i) f_0 and f_1 are regular homotopic;
- (ii) $f_0|_{N_0}$ and $f_1|_{N_0}$ are regular homotopic;
- (iii) $W'_0(f_1|_{N_0}, f_0|_{N_0}) = 0$;
- (iv) $\lambda(f_0)(x, x) \equiv \lambda(f_1)(x, x) \pmod{2}$ for each $x \in H_3$.

Indeed,

• (i) \Leftrightarrow (ii) because by the Smale-Hirsch classification of immersions [Hi60] the complete obstruction to extension of a regular homotopy from N_0 to N assumes values in $\pi_4(V_{7,4}) = 0$ [Pa56].

• (ii) \Leftrightarrow (iii) because by the Smale-Hirsch classification of immersions [Hi60] the first obstruction to regular homotopy between $f_0|_{N_0}$ and $f_1|_{N_0}$ assumes values in $H_1(N_0, \partial; \pi_3(V_{7,4}))$ and is complete, and because this obstruction clearly coincides with $W'_0(f_1|_{N_0}, f_0|_{N_0})$.

• (iii) \Leftrightarrow (iv) by Lemma 2.18 because by the calculation of λ (Lemma 1.3.c) $\lambda(f_k)([X], [X]) = \text{lk}(f_k|_X, f_k|_{X'})$, so $W'_0(f_0, f_1) \cap_N x = \rho_2(\lambda(f_0) - \lambda(f_1))(x, x)$ for all $x \in H_3(N; \mathbb{Z}_2)$. □

Problem 2.19 (Compression problem). For an integer $j \geq 1$, describe those embeddings $N \rightarrow S^7$ which are isotopic to embeddings with image in $S^{7-j} \subset S^7$.

Clearly, $\lambda(f) = \varkappa(f) = 0$ for an embedding $f : N \rightarrow \mathbb{R}^7$ such that $f(N) \subset \mathbb{R}^6$.

Proposition 2.20. *There are embeddings $f_0, f_1 : N \rightarrow S^7$ such that $f_0(N) \cup f_1(N) \subset S^6$ and $\beta(f_0, f_1) \neq 0$.*

Proof. This follows by Lemma 2.21 below because $\beta(\tau(0, 0), \tau(0, 1)) \neq 0$ by the calculation of β (Lemma 2.7.a). □

Lemma 2.21. *There is a representative of $\tau(0, 1)$ whose image is in $S^6 \subset S^7$.*

Remark 2.22 (An alternative construction of $\tau(1, 0)$ and $\tau(0, 1)$). The isotopy classes $\tau(1, 0)$ and $\tau(0, 1)$ are represented by embeddings

$$S^1 \times S^3 \xrightarrow{\text{pr}_2 \times T^k} S^3 \times S^3 \xrightarrow{i} S^7,$$

where the maps $T^k : S^1 \times S^3 \rightarrow S^3$ are defined as follows:

- $T^1(s, y) := sy$, where S^3 is identified with the set of unit length quaternions and $S^1 \subset S^3$ with the set of unit length complex numbers;
- $T^2(e^{i\theta}, y) := \eta(y) \cos \theta + \sin \theta$, where $\eta : S^3 \rightarrow S^2$ is the Hopf map and S^2 is identified with the 2-sphere formed by unit length quaternions of the form $ai + bj + ck$.

Proof of Lemma 2.21. We use the construction of $\tau(0, 1)$ from Remark 2.22. Denote by $n : S^2 \rightarrow TS^3$ a non-zero vector field normal to $S^2 \subset S^3$ and looking to the northern hemisphere of S^3 . Then

- for each $x \in S^3$ the image $T^2(S^1 \times x)$ is the round circle in S^3 passing through x in the direction $n(\eta(x))$, and
- T^2 is uniform on this circle.

Consider the normal bundle of $\text{id} : S^3 \times \eta : S^3 \rightarrow S^3 \times S^2$. The obstructions to the existence of a non-zero section of this bundle are in $H^{i+1}(S^3, \pi_i(S^1)) = 0$. Hence there is such a section $v(x) \in T_{\eta(x)}S^2$, $x \in S^3$. Define a map $T^3 : S^1 \times S^3 \rightarrow S^3$ by setting

- for each $x \in S^3$ the image $T^3(S^1 \times x)$ to be the round circle in S^3 passing through x in the direction $v(x)$, and

- T^3 to be ‘linear’ uniform on this circle.

We have $T^3(S^3) \subset S^2$, hence $i \circ (\text{pr}_2 \times T^3)(S^1 \times S^3) \subset i(S^3 \times S^2) \subset S^6 \subset S^7$.

Take a linear homotopy $v_t(x) := \frac{tn(\eta(x)) + (1-t)v(x)}{|tn(\eta(x)) + (1-t)v(x)|} \in T_{\eta(x)}S^3$ between non-zero vector fields $n(\eta(x))$ and $v(x)$ on $S^2 \subset S^3$. This homotopy defines a homotopy between T^2 and T^3 which keeps the image of $S^1 \times x$ embedded. The latter homotopy defines an isotopy from a representative $i \circ (\text{pr}_2 \times T^2)$ of $\tau(0, 1)$ to the embedding $i \circ (\text{pr}_2 \times T^3)$ whose image is in $S^6 \subset S^7$. \square

3 Proofs of lemmas

3.1 More notation

Recall that some notation was introduced in §§1.2, 1.4 and 2.1.

Denote by $D\nu = D\nu_f : S^7 - \text{Int } C_f \rightarrow N$ the oriented normal disk bundle of f (the orientation of $D\nu$ is inherited from the orientation of S^7 and N).

Definition of homological Alexander duality. Consider the following diagram:

$$\begin{array}{ccccc}
 & & H_{q-2}(N) & \xrightarrow{\text{PD}} & H^{6-q}(N) \\
 & & \downarrow \nu^! & \searrow \hat{A} & \parallel_{AD} \\
 H_{q+1}(C, \partial) & \xrightarrow{\partial_C} & H_q(\partial C) & \xrightarrow{i_C} & H_q(C) \\
 \parallel_{\text{PD}} & \swarrow A & \downarrow \nu & & \\
 H^{6-q}(C) & \xrightarrow{AD} & H_q(N) & &
 \end{array}$$

Here AD is Alexander duality and $A = A_f, \hat{A} = \hat{A}_f$ are homological Alexander duality isomorphisms. The lines are exact and the squares are commutative by the well-known Alexander Duality Lemmas of [Sk08’, Sk10].

Fix an orientation on N and denote by $[N] \in H_4$ and $[N_0] \in H_4(N_0, \partial)$ the corresponding fundamental classes. We often use the class $A[N] \in H_5(C, \partial)$ which may be called the *homology Seifert surface* of f .

Lemma 3.1 (Intersection Alexander duality). *For all $y \in H_q$ and for all $z \in H_{4-q}$ we have $y \cap_N z = Ay \cap_C \hat{A}z$.*

Proof. For all $x \in H_r(\partial C)$ we have $\nu(x \cap_{\partial C} \nu^! z) = \nu x \cap_N z$. Take $x = \partial Ay$. Since $\nu x = y$ and $y \cap_N z \in \mathbb{Z}$, we obtain $y \cap_N z = \partial Ay \cap_{\partial C} \nu^! z = Ay \cap_C \hat{A}z$. \square

Definition of the restriction homomorphism r . If P is a compact codimension c submanifold of a compact manifold Q and either $y \in H_k(Q)$ or $y \in H_k(Q, \partial)$, denote

$$r_{P,Q}(y) = r_P(y) = y \cap P := \text{PD}((\text{PD}y)|_P) \in H_{k-c}(P, \partial).$$

If y is represented by a closed oriented submanifold $Y \subset Q$ transverse to P , then $y \cap P$ is represented by $Y \cap P$. Clearly, $y \cap_Q [P] = i_{P,Q}(y \cap P)$.

Definition of the difference class $d(\xi, \xi')$. (This definition is not used until §3.3.) Let Q be a compact q -manifold, and ξ, ξ' non-zero sections of a k -dimensional vector bundle over Q . We define the *difference class*

$$d(\xi, \xi') \in H_{q-k+1}(Q, \partial) = H^{k-1}(Q) = H^{k-1}(Q; \pi_{k-1}(S^{k-1}))$$

of ξ and ξ' to be the class of the preimage of the zero section under a general position homotopy from ξ to ξ' . This class is the *homology primary obstruction* to a vertical homotopy from ξ to ξ' , and is equal to the Poincaré dual of the *cohomology primary obstruction* to a vertical homotopy from ξ to ξ' , which is defined in [Wh78, Theorem 6.4 Ch. VI].

Difference classes between other structures, e.g. spin structures or framings, on (a part of) a manifold are defined analogously. (In fact, such structures can be represented as sections of certain bundles. Then one can use the homological or cohomological definition of the primary obstruction to vertical homotopy, the two definitions being related by Poincaré duality.)

Definition of cobordism of homology classes together with supporting manifolds.

(This definition is not used until §3.8.) Assume that P and Q are closed oriented manifolds and $x_j \in H_{k_j}(P)$, $y_j \in H_{k_j}(Q)$ for $j = 1, \dots, n$. The tuples (P, x_1, \dots, x_n) and (Q, y_1, \dots, y_n) are called *cobordant* if there is a compact oriented manifold V and classes $v_j \in H_{k_j+1}(V, \partial)$ such that

$$\partial V = P \sqcup (-Q), \quad \partial v_j \cap P = x_j \quad \text{and} \quad \partial v_j \cap Q = y_j \quad \text{for each } j = 1, \dots, n.$$

The following definitions of a spin structure on a manifold Q and of the spin characteristic class p_Q^* will not be used until §3.6. Take a compact manifold Q and its triangulation. We write ‘skeleta of Q ’ for ‘skeleta of the triangulation’.

Definition of a spin structure. A *spin structure* (more precisely, *stable tangent spin structure*) on Q is a stable tangent framing over the 2-skeleton of Q . Two spin structures on Q are *equivalent* if their restrictions to the 1-skeleton are homotopic.

The trivial spin structure on S^7 is the one induced from the spin structure on D^8 compatible with the orientation.

If $P \subset Q$ is a compatibly triangulated codimension zero submanifold, then a spin structure s on Q *induces* a spin structure on P by restricting the stable framing on the 2-skeleton of Q to the 2-skeleton of P . If Q has boundary ∂Q , then s induces a spin structure on ∂Q .

If $F: Q \rightarrow P$ is a diffeomorphism with differential dF and s is a spin structure on Q , then the *induced* spin structure F_*s on P is obtained by applying dF to the vector fields over the 2-skeleton of Q which define s .

Remark on spin structures via maps to $BSpin$. Let BSO and $BSpin$ be the classifying spaces for stable oriented and stable spin vector bundles respectively. In other words, $BSpin = BO\langle 4 \rangle$ is the (unique up to homotopy) 3-connected space for which there exists a fibration $\gamma: BSpin \rightarrow BO$ inducing an isomorphism on π_i for each $i \geq 4$. Let $\tau_Q: Q \rightarrow BO$ the classifying map of the stable tangent bundle. A *spin lift* of τ_Q is a map $\bar{\tau}_Q: Q \rightarrow BSpin$ with $\tau_Q = \gamma \circ \bar{\tau}_Q$. Obstruction theory shows that a spin lift of τ_Q defines a framing of the stable tangent bundle of Q over the 2-skeleton. Obstruction theory also gives that two spin lifts of τ_Q are vertically homotopic over γ if and only if the corresponding framings are homotopic over the 1-skeleton. Hence a spin structure on Q may be regarded as a vertical homotopy class of a spin lift $\bar{\tau}_Q$ of τ_Q .

Definition of p_Q^* for a spin q -manifold Q . It is well-known that there is a generator $p \in H^4(BSpin) \cong \mathbb{Z}$ such that $2p$ is the pull back in $H^4(BSpin)$ of the universal first Pontryagin class $p_1 \in H^4(BSO)$ [CS11, §3, proof of Lemma 2.11]. Take the map $\bar{\tau}_Q: Q \rightarrow BSpin$ corresponding to the spin structure on Q and define

$$p_Q^* := \text{PD} \bar{\tau}_Q p \in H_{q-4}(Q, \partial).$$

We remark that p_Q^* does not depend of the choice of spin structure on Q [CCV08, page 170] (for simply-connected Q this is obvious).

3.2 Lemmas on the κ - and λ -invariants (3.2 and 2.2)

In this subsection

- the larger intersection symbol \bigcap denotes the intersection of homology classes in $\nu^{-1}N_0$;
- we identify $H_q(N_0, \partial)$ with H_q by the isomorphism $r_{N_0, N}$ for each $q \in \{1, 2, 3\}$;
- for a section $\xi: N_0 \rightarrow \nu^{-1}N_0$ we use without mention that $\xi[N_0] \bigcap \nu^!y = \xi y$ for each $q \in \{1, 2, 3\}$ and $y \in H_q$;
- we shorten $\lambda(f)$, $\overline{\lambda(f)}$ and $\varkappa(f)$ to λ , $\overline{\lambda}$ and \varkappa respectively;
- we define $\overline{\varkappa}: H_2 \rightarrow \mathbb{Z}$ by $\overline{\varkappa}(y) := \varkappa \cap y$;
- before we prove that λ and \varkappa are independent of ξ (Lemma 3.2. λ', \varkappa') we denote them by λ_ξ and \varkappa_ξ respectively.

Lemma 3.2. *Let $\xi: N_0 \rightarrow \nu^{-1}N_0$ be a section such that $i_C\xi$ is weakly unlinked.*

(a) $i_C(\xi[N_0] \bigcap x) = A[N] \cap_C i_Cx$ for each $q \in \{1, 2, 3\}$ and $x \in H_q(\nu^{-1}N_0)$.

(λ) $\overline{\lambda}_\xi = \widehat{A}^{-1}i_C\xi$ on H_3 .

(\varkappa) $\overline{\varkappa}_\xi = \widehat{A}^{-1}i_C\xi$ on H_2 .

(λ') $\overline{\lambda}_\xi(y) = \widehat{A}^{-1}(A[N] \cap_C \widehat{A}y)$ for each $y \in H_3$.

(\varkappa') $\varkappa_\xi = A^{-1}(A[N] \cap_C A[N])$.

(λ'') $\nu^!\overline{\lambda} = \xi - \partial A$ on H_3 .¹⁰

(\varkappa'') $\nu^!\overline{\varkappa} = \xi - \partial A$ on H_2 .

(e) $\varkappa = e^*(\xi^\perp)$, where ξ^\perp is the normal bundle of $\xi: N_0 \rightarrow \partial C$ (or, equivalently, the orthogonal complement to ξ in $D\nu|_{N_0}$).

Proof of (a). By [Sk10, Section Lemma 2.5.a] $A[N] \cap \nu^{-1}N_0 = \xi[N_0]$. Hence we have the equalities $A[N] \cap_C i_Cx = i_C((A[N] \cap \nu^{-1}N_0) \bigcap x) = i_C(\xi[N_0] \bigcap x)$. \square

Proof of (λ) and (\varkappa). The formulas follow because for each closed oriented q -submanifold $X \subset N$, $q \in \{3, 4\}$,

$$\text{lk}_{S^7}(fX, \xi Y) \stackrel{(1)}{=} \text{lk}_{S^7}(\partial AX, \xi' Y) \stackrel{(2)}{=} A[X] \cap_C i_C \xi y \stackrel{(3)}{=} [X] \cap_N \widehat{A}^{-1}i_C \xi y$$

where

- AX is any $(q+1)$ -chain in C whose boundary is in ∂C and represents $\partial A[X]$ there;
- $\xi' Y$ is a small shift of ξY into the interior of C ;
- (1) holds because $\nu \partial A[X] = [X]$, so fX is homologous to ∂AX in $S^7 - \text{Int } C$, and then because ξY is homologous to $\xi' Y$ in C ;
- (2) holds by definition of the linking coefficient;
- (3) holds by intersection Alexander duality (Lemma 3.1). \square

Proof of (λ') and (\varkappa'). By (a) for $x = \nu^!y$ we have $A[N] \cap_C \widehat{A}y = i_C(\xi[N_0] \bigcap \nu^!y) = i_C \xi y \in H_q(C)$ for each $q \in \{2, 3\}$. So (λ) implies (λ'). Also (\varkappa) implies that for each $y \in H_2$ we have

$$\varkappa \cap_N y = \widehat{A}^{-1}(A[N] \cap_C \widehat{A}y) = A[N] \cap_C (A[N] \cap_C \widehat{A}y) = A^{-1}(A[N] \cap_C A[N]) \cap_N y.$$

Here the second and the third equalities follow by intersection Alexander duality (Lemma 3.1). This proves (\varkappa'). \square

Proof of (e). Part (e) follows because for all $y \in H_2$

$$\varkappa \cap_N y \stackrel{(1)}{=} A[N] \cap_C i_C \xi y \stackrel{(2)}{=} \xi[N_0] \bigcap \xi y = \xi[N_0] \bigcap \xi[N_0] \bigcap \nu^!y = e^*(\xi^\perp) \cap_N y.$$

Here (1) holds by (\varkappa) and intersection Alexander duality (Lemma 3.1), (2) holds by (a) and the other two equalities are obvious. \square

¹⁰Parts (λ'') and (\varkappa'') are only used in the alternative proof of Lemma 3.5.a,b.

Proof of (λ'') and (\varkappa'') . Let $q \in \{2, 3\}$. Since $\nu\partial A = \text{id } H_q = \nu\xi$, for all $y \in H_q$ there is a class $x(y) \in H_{q-2}$ such that $\partial Ay - \xi y = \nu^!x(y)$. Then $0 = i_C\partial Ay = i_C\xi y + i_C\nu^!x(y) = i_C\xi y + \widehat{A}x(y)$. Now (λ'') and (\varkappa'') follow by (λ) and (\varkappa) . \square

Proof of \varkappa -symmetry (Lemma 2.2). Let $\xi: N_0 \rightarrow \partial C$ be a weakly unlinked section obtained from a section $\zeta: N_0 \rightarrow \nu^{-1}N_0$ by composing with the inclusion $\nu^{-1}N_0 \rightarrow \partial C$. Let $-\xi: N_0 \rightarrow \partial C$ be the weakly unlinked section obtained by composing $-\zeta$ with the inclusion $N_0 \rightarrow \partial C$. If the homology classes $x, y \in H_3$ are represented by closed oriented 3-submanifolds (or integer 3-cycles) $X, Y \subset N_0$, then

$$\lambda(y, x) = \text{lk}_{S^7}(fY, \xi X) = \text{lk}_{S^7}(\xi X, fY) = \text{lk}_{S^7}(fX, -\xi Y).$$

Hence

$$\begin{aligned} \lambda(x, y) - \lambda(y, x) &\stackrel{(1)}{=} fX \cap_{S^7} Y_\xi \stackrel{(2)}{=} \xi X \cap_{S^7} Y'_\xi \stackrel{(3)}{=} \xi x \cap_{\partial C} \xi y \stackrel{(4)}{=} \\ &= \xi[N_0] \bigcap \nu^!x \bigcap \xi[N_0] \bigcap \nu^!y \stackrel{(5)}{=} \xi[N_0] \bigcap \xi[N_0] \bigcap \nu^!(x \cap_N y) \stackrel{(6)}{=} \\ &= \xi[N_0] \bigcap \xi(x \cap_N y) \stackrel{(7)}{=} A[N] \cap_C i_C \xi(x \cap_N y) \stackrel{(8)}{=} \varkappa \cap_N x \cap_N y, \end{aligned}$$

where

- $Y_\xi \subset S^7$ is the 4-submanifold (with boundary) that is the union over $a \in Y$ of segments joining ξa to $(-\xi)a$ (or Y_ξ is the corresponding integer 4-chain); we have $Y_\xi \cong Y \times I$;
- the algebraic intersection of submanifolds (or the cycle and the chain) in S^7 is defined because the first one does not intersect the boundary of the second one;
- (1) holds by definition of the linking coefficient and the above formula for $\lambda(y, x)$;
- Y'_ξ is obtained from Y_ξ by a small shift along $\xi - f$ considered as vector field on fN ;
- (2), (4), (6) are clear;
- (3) holds because $Y'_\xi \cap \partial C = \xi Y$;
- (5) holds because $\dim \nu^{-1}N_0 - \dim \xi[N_0]$ is even, so we can exchange the order of terms in the cap product without changing the sign, and because $\nu^!x \bigcap \nu^!y = \nu^!(x \cap_N y)$;
- (7) holds by Lemma 3.2.a;
- (8) holds by Lemma 3.2. \varkappa and intersection Alexander duality (Lemma 3.1). \square

Remarks. (a) The class $\varkappa \in H_2$ measures the linking of 2-cycles in N and the ‘top cell’ of N under f : if $f = f'$ on N_0 , then $(\varkappa(f') - \varkappa(f)) \cap N_0 = 2\widehat{A}_{f|N_0}^{-1}[f(B^4) \cup f'(B^4)] \in H_2(N_0; \mathbb{Z}) \cong H_2$ (this is proved analogously to [Sk08', §2, The Boéchat-Haefliger invariant Lemma]).

(b) Weakly unlinked sections may differ on a 3-skeleton of N_0 even up to vertical homotopy, and *a priori* changing ξ on a 3-skeleton could change the integer $\text{lk}_{S^7}(fX, \xi Y)$ in the definition of λ . However different choices of ξ do not change $\text{lk}_{S^7}(fX, \xi Y)$. The formal explanation for this is given in Lemma 3.2. λ' . Informally, the change is trivial because it ‘factors through’ $H_3(S^2) = 0$.

(c) If in the definitions of \varkappa and λ we would take an arbitrary (i.e. not weakly unlinked) section ξ , we would obtain different values. Note that $2\lambda(x, y) = \text{lk}_{S^7}(fx, \xi y) + \text{lk}_{S^7}(fx, -\xi y)$ for any (i.e., not necessarily weakly unlinked) section ξ (D. Tonkonog, unpublished, cf. [To10]).

(d) Although a weakly unlinked section is only defined over N_0 , its construction involves all of the embedding f via the inclusions $\nu^{-1}N_0 \rightarrow \partial C \rightarrow C$, and not only $f|_{N_0}$. For embeddings $N_0 \rightarrow S^7$ an analogue of \varkappa is not defined and only ‘a part’ of λ is defined (D. Tonkonog, unpublished, cf. [To10]).

3.3 Parametric additivity of \varkappa and λ (Lemma 2.13)

Proof of Lemma 2.11. Define

$$\mathbf{i} : \sqrt{2}D^2 \times D^4 \rightarrow S^7 \quad \text{by} \quad \mathbf{i}(x, y) := (y\sqrt{2 - |x|^2}, 0, 0, x)/\sqrt{2}.$$

Then $\mathbf{i} = \tau_0$ on $S^1 \times D^3$. For $\gamma \leq \sqrt{2}$ denote $\Delta_\gamma := \mathbf{i}(\gamma D^2 \times \{-1_3\}) \subset \text{Int } D_-^7$.

In this proof we omit the sign \circ for composition.

Any two embeddings $S^1 \times D^3 \rightarrow S^7$ are isotopic. So we can make an isotopy and assume that $f s = \mathbf{i}$ on $S^1 \times D_-^3$.

Since $7 > 2 \cdot 1 + 3 + 1$, by general position we may assume that $f(N) \cap \Delta_1 = \partial\Delta_1$. Then there is γ slightly greater than 1 such that $f(N) \cap \Delta_\gamma = \partial\Delta_1$. Take the standard 3-framing on Δ_γ tangent to $\mathbf{i}(\gamma D^2 \times S^3)$ whose restriction to $\partial\Delta_1$ is the standard normal 3-framing of $\partial\Delta_1$ in $f(N)$. Then the standard 2-framing normal to $\mathbf{i}(\gamma D^2 \times S^3)$ is a 2-framing on $\partial\Delta_1$ normal to $f(N)$. Using these framings we construct

- an orientation-preserving embedding $H : D_-^7 \rightarrow D_-^7$ onto a sufficiently small neighborhood of Δ_1 in D_-^7 , and
- an isotopy h_t of $\text{id}(S^1 \times S^3)$ shrinking $S^1 \times D_-^3$ to a sufficiently small neighborhood of $S^1 \times \{-1_3\}$ in $S^1 \times D_-^3$ such that

$$H(\Delta_{\sqrt{2}}) = \Delta_\gamma, \quad H \mathbf{i}(S^1 \times D_-^3) = H(D_-^7) \cap f(N) \quad \text{and} \quad H \mathbf{i} = \mathbf{i} h_1 \quad \text{on} \quad S^1 \times D_-^3.$$

The embedding H is isotopic to $\text{id } D_-^7$ by [Hi76, Theorem 3.2]. This isotopy extends to an isotopy H_t of $\text{id } S^7$ by the Isotopy Extension Theorem [Hi76, Theorem 1.3]. Then $H_t^{-1} f h_t$ is an isotopy of f . Let us prove that the embedding $H_1^{-1} f h_1$ is standardized.

We have $H_1^{-1} f h_1 = H_1^{-1} \mathbf{i} h_1 = \mathbf{i}$ on $S^1 \times D_-^3$. Also if $H_1^{-1} f h_1(N - \text{im } s) \not\subset \text{Int } D_+^m$, then there is $x \in N - \text{im } s$ such that $f h_1(x) \in H(D_-^7)$. Then $f h_1(x) = H \mathbf{i}(y) = \mathbf{i} h_1(y) = f h_1(y)$ for some $y \in S^1 \times D_-^3$. This contradicts the fact that $f h_1$ is an embedding. \square

Let V be a compact oriented 4-manifold with non-empty boundary. Recall that an embedding $v : V \rightarrow D_+^7$ is called *proper*, if $f^{-1} \partial D_+^7 = \partial V$. Denote by

- $C = C_v$ the closure of the complement in D_+^7 to a tubular neighborhood of $v(V)$;
- $\nu = \nu_v : \text{Cl}(\partial C_v - \partial D_+^7) \rightarrow V$ the restriction of the oriented normal vector bundle of v .

A section $\zeta : V \rightarrow \nu^{-1}V = \text{Cl}(\partial C_v - \partial D_+^7)$ of ν is called *weakly unlinked* if

$$i_{\partial C, C} \zeta[V] = 0 \in H_4(C, \partial D_+^7 \cap \partial C).$$

We remark that if we would take $\partial V = \emptyset$ in this definition, we would not obtain the definition of a weakly unlinked section for a closed manifold.

Lemma 3.3. (a) Any section is weakly unlinked for any proper embedding $v : D^1 \times S^3 \rightarrow D_+^7$.

(b) For any proper embedding $v : V \rightarrow D_+^7$ of a compact connected oriented 4-manifold V with non-empty boundary and torsion free $H_1(V, \partial)$, a weakly unlinked section exists and is unique up to vertical homotopy over any 2-skeleton of V .

(c) Let $g : N \rightarrow S^7$ and $s : D^1 \times S^3 \rightarrow N$ be embeddings such that $g|_{g^{-1}(D_\pm^7)}$ is a proper embedding into D_\pm^7 and $g^{-1}(D_-^7) = \text{im } s$. Then any weakly unlinked section for the abbreviation of g ,¹¹ $N - \text{Int}(\text{im } s) \rightarrow D_+^7$, extends over $N_0 := N - \text{Int } s(D^1 \times D_-^3)$ to a weakly unlinked section for g .

Proof of (a). Part (a) follows because

$$H_4(C_v, \partial D_+^7 \cap \partial C_v) \cong H_4(D^7 - \mathbf{i}(D^1 \times S^3), \partial D_+^7 - \mathbf{i}(S^0 \times S^3)) \cong H_2(D^1 \times S^3, \partial) = 0$$

by Alexander duality, the homology exact sequence of a pair and the 5-lemma. (Cf. the proof of additivity of β , Lemma 2.9, in §3.7.) \square

¹¹For a map $f : X \rightarrow Y$ and $A \subset X$, $f(A) \subset B \subset Y$, the *abbreviation* $g : A \rightarrow B$ of f is defined by $g(x) := f(x)$.

Proof of (b). (The proof is analogous to the ‘absolute’ case [BH70, Proposition 1.3].) For sections ξ and ξ' of the normal bundle of v the *difference class* $d(\xi, \xi') \in H_2(V, \partial)$ is defined in §3.1. Alexander duality $\widehat{A}: H_q(V, \partial) \rightarrow H_{q+2}(C_v, \partial D_+^7 \cap \partial C)$ is defined analogously to the absolute case. Then $d(\xi, \xi') = \pm \widehat{A}^{-1}(\xi - \xi')[V, \partial V]$ analogously to [BH70, Lemme 1.2]. This implies the uniqueness of a weakly unlinked section for v .

Let us prove the existence of a weakly unlinked section for v . The normal Euler class $\bar{e}(v)$ assumes values in $H^3(V) \cong H_1(V, \partial)$. Since the normal bundle of v is odd-dimensional, $2\bar{e}(v) = 0$.¹² Since $H_1(V, \partial)$ is torsion free, $\bar{e}(v) = 0$. Since V is connected and has non-empty boundary, it retracts to a 3-dimensional subpolyhedron. Hence there is a section $\zeta: V \rightarrow \nu^{-1}V$ of ν . Denote by U a closed neighbourhood of a 2-skeleton in V . Construct a section $\xi': U \rightarrow \nu^{-1}V$ such that $d(\xi', \zeta|_U) = \mp \widehat{A}^{-1}\zeta[V] \cap U \in H_2(U, \partial)$. By [St99, Theorem 37.4] there is an extension ξ of ξ' to V such that $d(\xi, \zeta) = \mp \widehat{A}^{-1}\zeta[V]$. Since $d(\xi, \zeta) = \pm \widehat{A}^{-1}(\xi - \zeta)[V]$, the extension ξ is weakly unlinked. \square

Proof of (c). Since $H^q(D^1 \times D_-^3, \partial D^1 \times D_-^3; \pi_{q-1}(S^2)) = 0$ for each q , obstruction theory entails that there is a section $\xi: N_0 \rightarrow \partial C_g$ extending a given weakly unlinked section for the abbreviation of g . Define $x \in H_4(C_g)$ by $x := i_{C_g} j_{\partial C_g}^{-1} \text{ex}^{-1} \xi[N_0, \partial]$, as in the definition of a weakly unlinked section for closed manifolds (§2.2). We have

$$x \cap D_+^7 = \xi[g^{-1}(D_+^7)] = 0 \in H_4(C_g \cap D_+^7, C_g \cap \partial D_+^7).$$

Consider the following part of the homology exact sequence of the pair $(C_f, C_f \cap \partial D_-^7)$:

$$\begin{array}{ccccc} H_4(C_g \cap D_-^7) & \longrightarrow & H_4(C_g) & \xrightarrow{j} & H_4(C_g, C_g \cap D_-^7) \\ \parallel \widehat{A}_g & & & & \parallel \text{ex}_+ \\ H_2(D^1 \times S^3) = 0 & & & & H_4(C_g \cap D_+^7, C_g \cap \partial D_+^7) \end{array}$$

Since $0 = x \cap D_+^7 = \text{ex}_+ jx$, we have $x = 0$, i.e. ξ is weakly unlinked for g . \square

Some notation for the proof of parametric additivity. Recall that $s: S^1 \times D^3 \rightarrow N$ is an embedding realizing $[s] \in H_1$ and R is the symmetry of S^m with respect to the subspace defined by $x_1 = x_2 = 0$. Let $N_+ := \text{Cl}(N - \text{im } s)$ and $N_- := \text{im } s \subset N$. For a cycle X representing $[X] \in H_l$ and in general position to $N_+ \cap N_-$ denote $X_\pm := X \cap N_\pm$ a relative cycle representing $[X] \cap N_\pm \in H_l(N_\pm, \partial)$.

By Lemma 2.11 we may assume that f and an embedding τ_α representing $\tau(l, b)$ are s -standardized and i -standardized, respectively. Take the embedding h given in the definition of parametric connected sum in §2.4. Then both f and h satisfy the assumptions of Lemma 3.3.c. We have $f = h$ on N_+ . Using Lemma 3.3.abc we can form a weakly unlinked section ξ_h for h as follows: we take the union of

- a weakly unlinked section for the abbreviation $N_+ \rightarrow D_+^7$ of h with
- the restriction to $s(D_+^1 \times D_-^3)$ of a weakly unlinked section for the abbreviation $N_- \rightarrow D_-^7$ of h .

A weakly unlinked section ξ_f for f can be constructed analogously: simply replace h by f .

Completion of the proof of parametric additivity for \varkappa . For each $x \in H_2$ take an integer 2-cycle (or closed oriented 2-submanifold) $X \subset N$ representing x . By general position we may assume that $X \subset N_+$. There is an integer 3-chain X' in D_+^7 such that $\partial X' = \xi_h X$. So parametric additivity for \varkappa holds because

$$\varkappa(h) \cap_N x = \text{lk}_{S^7}(hN, \xi_h X) = hN \cap_{S^7} X' \stackrel{(3)}{=} fN \cap_{S^7} X' \stackrel{(4)}{=} \varkappa(f) \cap_N x,$$

¹²Alternatively, since $\bar{e} = 0$ for embeddings of closed manifolds, $2\bar{e}(v) = \bar{e}(2v) = 0$, where $2v$ is the ‘double’ of v .

where

- the equality (3) follows because $h = f$ on N_+ and $hN_-, fN_- \subset D_-^7$;
- the equality (4) is proved in the same ways as the first two equalities, with h replaced by f . \square

Completion of the proof of parametric additivity for λ . For all $x, y \in H_3$ take integer 3-cycles (or closed oriented 3-submanifolds) $X, Y \subset N$ representing x, y . There are integer 4-chains Y'_\pm in D_\pm^7 such that $\partial(Y'_+ + Y'_-) = \xi_h Y$ and $\partial Y'_\pm \cap hN = \emptyset$. We have

$$\lambda(h)(x, y) \stackrel{(a)}{=} \text{lk}_{S^7}(hX, \xi_h Y) = hX_+ \cap_{S^7} Y'_+ + hX_- \cap_{S^7} Y'_-.$$

So parametric additivity for λ follows because

$$\begin{aligned} hX_+ \cap_{S^7} Y'_+ &\stackrel{(*)}{=} fX_+ \cap_{S^7} Y'_+ \stackrel{(**)}{=} \lambda(f)(x, y) \quad \text{and} \\ hX_- \cap_{S^7} Y'_- &\stackrel{(1)}{=} \tau_\alpha s^{-1} X_- \cap_{S^7} RY'_- \stackrel{(2)}{=} (\lambda\tau_\alpha)(x_s, y_s) \stackrel{(3)}{=} l([s] \cap_N x)([s] \cap_N y). \end{aligned}$$

Here

- equality $(*)$ holds because $h = f$ on N_+ and $hN_-, fN_- \subset D_-^7$;
- equality $(**)$ holds by equality (a) for h replaced by f , because $fs = i|_{S^1 \times D_-^3}$, so $fX_- \cap_{S^7} Y'_- = 0$ analogously to the calculation of λ (Lemma 1.3.c);
- equality (1) holds because R preserves the orientation;
- $x_s := ([s] \cap_N x)[1_1 \times S^3] \in H_3(S^1 \times S^3)$ and analogously define y_s ;
- equality (2) is proved below;
- equality (3) holds by the calculation of λ (Lemma 1.3.b).

To prove equality (2), we first apply the analogue of equality $(**)$ for f replaced by τ_α . Observe that $R\xi_s$ is a weakly unlinked section for τ_α . This shows that the left hand side of equality (2) equals to the value of $\lambda\tau_\alpha$ on certain homology classes in $H_3(S^1 \times S^3)$. We have the equality $[s^{-1}X_-] = ([s] \cap_N x)[1_1 \times D_+^3] = x_s \cap (1_1 \times D_+^3)$ and the same with X, x replaced by Y, y . Hence these homology classes are x_s and y_s . \square

Remark. The calculation of λ (Lemma 1.3.b) follows by Remark 2.12 and the parametric additivity (Lemma 2.13). However, Lemma 2.13 for λ is proved using Lemma 1.3.b. For this reason, as well as for applications, it is convenient to state Lemma 1.3.b separately.

3.4 Agreement of Seifert classes (Lemmas 2.4 and 3.5)

Lemma 3.4. *If $\varkappa(f_0) = \varkappa(f_1)$ and $\lambda(f_0) = \lambda(f_1)$, $\varphi : \partial C_0 \rightarrow \partial C_1$ is a bundle isomorphism and $\xi : N_0 \rightarrow \partial C_0$ a weakly unlinked section for f_0 , then $\varphi\xi$ is a weakly unlinked section for f_1 .*

Proof. The proof we give follows the same line of reasoning as [CS11, §3, end of proof of Lemma 2.5]. By Lemma 2.1 there exists a weakly unlinked section ξ_1 for f_1 . By Lemma 3.2.e

$$e^*((\varphi\xi)^\perp) = e^*(\xi^\perp) = \varkappa(f_0) = \varkappa(f_1) = e^*(\xi_1^\perp).$$

For any pair of sections $\zeta, \eta : N_0 \rightarrow \partial C_1$ we have

$$e^*(\zeta^\perp) - e^*(\eta^\perp) = \pm 2d(\zeta, \eta) = \pm 2i_{Cj_{\partial C}^{-1}} \text{ex}^{-1}(\zeta - \eta),$$

where $d(\zeta, \eta) \in H_2(N_0; \pi_2(S^2))$ is the difference class [BH70, Lemme 1.7], defined in §3.1 above. We apply this for $\zeta = \xi_1$ and $\eta = \varphi\xi$. Since $H_2(N)$ has no 2-torsion, we obtain the equation $i_{Cj_{\partial C}^{-1}} \text{ex}^{-1} \varphi\xi = i_{Cj_{\partial C}^{-1}} \text{ex}^{-1} \xi_1 = 0$; i.e. $\varphi\xi$ is weakly unlinked. \square

Definition of a joint Seifert class. A joint Seifert class for $x \in H_q$ and a bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ is an element

$$X \in H_{q+1}(M_\varphi) \quad \text{such that} \quad X \cap C_k = A_k x \in H_{q+1}(C_k, \partial) \quad \text{for each} \quad k = 0, 1.$$

When the bundle isomorphism φ is clear from the context, we shall simply call X a joint Seifert class for $x \in H_q$. Note that a joint Seifert class, as defined in §2.3, is a joint Seifert class for $[N] \in H_4$ by Lemma 3.13.a below.

Lemma 3.5 (Agreement of Seifert classes). *Assume that $\varkappa(f_0) = \varkappa(f_1)$, that $\lambda(f_0) = \lambda(f_1)$,¹³ and that $\varphi : \partial C_0 \rightarrow \partial C_1$ a bundle isomorphism. Assume that the coefficients are \mathbb{Z} or \mathbb{Z}_d for some d ; they are omitted from the notation. Let $\partial_k := \partial_{\partial C_k, C_k}$.*

- (a) $\varphi \partial_0 A_0 = \partial_1 A_1 : H_q \rightarrow H_q(\partial C_1)$.
- (b) $\partial : H_{q+1}(M_\varphi, C_0) \rightarrow H_q(C_0)$ is zero for each q .
- (c) For each $x \in H_q$ there is a joint Seifert class for x .

Proof. Part (c) follows by (a) and the Mayer-Vietoris sequence for M_φ :

$$H_5(M_\varphi) \xrightarrow{r_0 \oplus r_1} H_5(C_0, \partial) \oplus H_5(C_1, \partial) \xrightarrow{\partial_0 - \partial_1} H_4(\partial C_0).$$

For $q \geq 5$ part (a) is trivial and part (b) follows because $H_{q+1}(M_\varphi, C_0) \cong H_q = 0$.

For $q = 1$ part (b) is trivial and part (a) follows because $\varphi \partial_0 A_0 = \partial_1 A_1 = \nu_0^{-1}$.

Now assume that $q \in \{2, 3, 4\}$. Let $\xi_0 : N_0 \rightarrow \partial C_0$ be a weakly unlinked section for f_0 . Since $\varkappa(f_0) = \varkappa(f_1)$, $\lambda(f_0) = \lambda(f_1)$ and H_2 has no 2-torsion, by Lemma 3.4 $\xi_1 := \varphi \xi_0$ is a weakly unlinked section for f_1 . In this proof $k \in \{0, 1\}$. The map

$$\xi_k \oplus \nu_k^! : H_q \oplus H_{q-2} \rightarrow H_q(\partial C_k)$$

is an isomorphism for each $q \in \{2, 3\}$. The map

$$(j_k^{-1} \text{ex}_k^{-1} \xi_k) \oplus \nu_k^! : H_4(N_0, \partial) \oplus H_2 \rightarrow H_4(\partial C_k)$$

is an isomorphism. Let $i_k := i_{\partial C_k, C_k}$. We have $i_k \nu_k^! = \widehat{A}_k$. By Lemma 3.2.λ, \varkappa and [Sk10, Lemma 2.5.b] we have

$$i_k \xi_k = \widehat{A}_k \overline{\varkappa(f_k)} \quad \text{on} \quad H_2, \quad i_k \xi_k = \widehat{A}_k \overline{\lambda(f_k)} \quad \text{on} \quad H_3 \quad \text{and} \quad i_k j_k^{-1} \text{ex}_k^{-1} \xi_k [N_0] = i_k \partial_k A_k [N] = 0.$$

Hence $\varphi \ker i_0 = \ker i_1 = \text{im } \partial_1$. Then the following commutative diagram

$$\begin{array}{ccccc} H_{q+1}(M_\varphi, C_0) & \xrightarrow{\partial} & H_q(C_0) & & \\ \parallel \text{ex} & & \uparrow i_0 \varphi^{-1} & & \\ H_q & \xrightarrow{A_1} & H_{q+1}(C_1, \partial) & \xrightarrow{\partial_1} & H_q(\partial C_1) \xrightarrow{i_1} H_q(C_1), \end{array}$$

shows that $i_0 \varphi^{-1} \partial_1 = 0$, which implies (b).

Since $\nu_1 \varphi \partial_0 A_0 = \nu_0 \partial_0 A_0 = \text{id } H_q = \nu_1 \partial_1 A_1$, we have $\varphi \partial_0 A_0 - \partial_1 A_1 = \nu_1^! y$ for some map $y : H_q \rightarrow H_{q-2}$. Applying i_1 to both sides and using that $\varphi \partial_0 A_0 x \in \varphi \ker i_0 = \ker i_1$ we obtain $0 = \widehat{A}_1 y$. Hence $y = 0$, i.e. (a) holds. \square

Proof of Lemma 2.4. By Lemma 3.5.b, i_{C_0, M_φ} is injective. Since $H_2(M_\varphi, C_0) \cong H_2(C_1, \partial) \cong H_1$ is torsion free, i_{C_0, M_φ} is split injective. As $S_{f_0}^2 \in H_2(C_0) \cong \mathbb{Z}$ is primitive, $i_{C_0, M_\varphi} S_{f_0}^2 \in H_2(M_\varphi)$ is primitive. So by Poincaré duality there is a joint Seifert class $Y \in H_5(M_\varphi)$. \square

¹³Of these assumptions we need none for (a,b) and $q \notin \{2, 3, 4\}$, and only $\varkappa(f_0) = \varkappa(f_1)$ for (a,b) and $q \in \{2, 4\}$.

Remark. Lemma 2.4 also follows from Lemmas 3.5.c and 3.13.a.

An alternative proof of the Agreement Lemma 3.5.a,b for $q \in \{2, 3, 4\}$. We generalize the proof of [CS11, Agreement Lemma 2.5] which is part (a) for $q = 4$.

Let $\xi_0 : N_0 \rightarrow \partial C_0$ be a weakly unlinked section for f_0 . Since $\varkappa(f_0) = \varkappa(f_1)$, $\lambda(f_0) = \lambda(f_1)$ and H_2 has no 2-torsion, by Lemma 3.4 $\xi_1 := \varphi \xi_0$ is a weakly unlinked section for f_1 .

We have $\nu_1 \varphi \partial_0 A_0 = \nu_0 \partial_0 A_0 = \text{id } H_q = \nu_1 \partial_1 A_1$. Also

$$\xi_1^! \varphi \partial_0 A_0 = \xi_0^! \partial_0 A_0 \stackrel{(2)}{=} \left\{ \begin{array}{ll} \overline{\varkappa}(f_0) & q = 4 \\ \xi_0^! \xi_0 - \xi_0^! \nu_0^! \overline{\lambda}(f_0) = \xi_0^! \xi_0 - \overline{\lambda}(f_0) & q = 3 \\ \xi_0^! \xi_0 - \xi_0^! \nu_0^! \overline{\varkappa}(f_0) = \xi_0^! \xi_0 - \overline{\varkappa}(f_0) & q = 2 \end{array} \right\} \stackrel{(3)}{=} \xi_1^! \partial_1 A_1.$$

Here

- (2) holds because $\xi_0^! \partial_0 A_0[N] = \varkappa$ [Sk10, Lemma 2.5.b] and by Lemma 3.2. λ'', \varkappa'' ;
- (3) holds because $\xi_0^! \xi_0 = \xi_0^! \varphi^{-1} \varphi \xi_0 = \xi_1^! \xi_1$.

Now (a) follows because the map $\nu_1 \oplus \xi_1^! : H_3(\partial C_1) \rightarrow H_3 \oplus H_1$ is an isomorphism.

Let $i_k := i_{\partial C_k, C_k}$ and consider the diagram from the previous proof of (b). Part (b) follows from (a) because

$$\begin{aligned} (a) \quad &\Rightarrow \quad \varphi \text{im}(\partial_0 A_0) = \text{im}(\partial_1 A_1) \quad \Leftrightarrow \quad \varphi \text{im } \partial_0 = \text{im } \partial_1 \quad \Leftrightarrow \quad \varphi \ker i_0 = \ker i_1 \quad \Rightarrow \\ &\Rightarrow \quad \varphi \ker i_0 \supset \text{im } \partial_1 \quad \Leftrightarrow \quad i_0 \varphi^{-1} \partial_1 = 0 \quad \Leftrightarrow \quad (b). \end{aligned}$$

(In fact, (a) \Leftrightarrow $\varphi \ker i_0 = \ker i_1$, see the previous proof.) □

3.5 Spin bundle isomorphisms (Lemma 3.6)

Definition of sp_k and a spin bundle isomorphism. For $k = 0, 1$ let sp_k be the stable tangent spin structure on ∂C_k induced from the trivial spin structure on S^7 .

A bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ is called *spin* if it carries sp_0 to sp_1 .

Lemma 3.6 (Spin Lemma). (a) *For a bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ the manifold M_φ is spin if and only if φ is spin. Moreover, for each spin bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$, there is a unique spin structure on M_φ whose restrictions to C_0, C_1 are induced from S^7 .*

(b) *A spin bundle isomorphism exists and is unique (up to homotopy) over the 2-skeleton of any triangulation of N .*

Remark. (a) The ‘if’ part of the Spin Lemma 3.6.a can be proved as follows: If φ is spin, then the spin structures on C_0, C_1 coming from S^7 agree up to homotopy on the boundaries. Hence they can be glued together to give a spin structure on M_φ .

Below we give another proof together with the proof of the ‘only if’ and the ‘moreover’ parts.

(b) In order to illustrate the main idea of the Spin Lemma 3.6.b let us sketch of a proof for $N = S^1 \times S^3$. (The sketch is not formally used in the proof.) Take a smooth map $\alpha : S^1 \rightarrow SO_3$ representing the generator of $\pi_1(SO_3)$. Identify ∂C_f and $S^1 \times S^3 \times S^2$. Define a bundle automorphism f_α of $S^1 \times S^3 \times S^2$ by the formula $f_\alpha(x, y, z) = (x, y, \alpha(x)z)$. The manifold ∂C_f has precisely two equivalence classes of stable tangential spin structure and the self-bundle-isomorphism f_α acts by exchanging these. This implies the existence. The uniqueness follows from the fact that every spin bundle isomorphism is isotopic to f_α or the identity.

Definition of the difference class $d(\text{sp}, \text{sp}')$. Let Q be a compact q -manifold. For spin structures sp and sp' on Q their *difference* in

$$H_{q-1}(Q, \partial; \mathbb{Z}_2) = H^1(Q; \mathbb{Z}_2) = H^1(Q; \pi_1(SO))$$

is the primary obstruction to homotopy from sp to sp' , cf. §3.1. (This is the homology class represented by the degeneracy set of a general position homotopy, through ordered $(q-1)$ -sets of vectors, from sp to sp' .)

The following facts about spin structures are well known, follow by elementary obstruction theory, and will be used without mention:

- if the difference of sp and sp' is zero, then sp and sp' are equivalent, and
- for a compact q -manifold Q the difference with a fixed spin structure is a 1–1 correspondence between $H_{q-1}(Q, \partial; \mathbb{Z}_2) = H^1(Q; \mathbb{Z}_2)$ and spin structures on Q up to equivalence.

For spin structures sp and sp' on ∂C_1 let

$$d(\text{sp}, \text{sp}') \in H_3(N; \mathbb{Z}_2)$$

be the preimage of the difference class in $H_5(\partial C_1; \mathbb{Z}_2)$ under the isomorphism $\nu^!$.

Proof of the Spin Lemma 3.6.a. We have

$$\varphi \text{ is spin} \iff d(\varphi_*\text{sp}_0, \text{sp}_1) = 0 \iff w_2^*(M_\varphi) = 0 \iff M_\varphi \text{ is spin.}$$

Here the second equivalence holds because

- $w_2^*(M_\varphi) = i_{C_0, M_\varphi} \widehat{A}_0 d(\varphi_*\text{sp}_0, \text{sp}_1)$ by the naturality of the primary obstruction (the details are analogous to Lemma 3.10 below), and
- $H_6(M_\varphi, C_0) \cong H_5 = 0$, so i_{C_0, M_φ} is injective (cf. the Agreement Lemma 3.5.b for $s = 5$).

Let us prove the ‘moreover’ part.

Existence follows by the proof of the ‘if’ part above.

Let us prove uniqueness. The Mayer-Vietoris sequence for $M_\varphi = C_0 \cup (-C_1)$ gives that the sum of the restriction homomorphisms $H^1(M_\varphi; \mathbb{Z}_2) \rightarrow H^1(C_0; \mathbb{Z}_2) \oplus H^1(C_1; \mathbb{Z}_2)$ is injective. So a spin structure on M_φ is determined up to equivalence by its restrictions to C_0 and C_1 . Hence the required spin structure is unique. \square

For the proof of the Spin Lemma 3.6.b we need the following definitions and lemmas.

For bundle isomorphisms $\varphi, \psi: \partial C_0 \rightarrow \partial C_1$ their *difference*

$$d(\varphi, \psi) \in H^1(N; \pi_1(SO_3)) = H_3(N; \mathbb{Z}_2)$$

is the primary obstruction to homotopy of bundle isomorphisms between φ and ψ .

Lemma 3.7. (a) For every bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$ and $v \in H_3(N; \mathbb{Z}_2)$ there is a bundle isomorphism $\varphi_v: \partial C_0 \rightarrow \partial C_1$ such that $d(\varphi_v, \varphi) = v$.

(b) For each pair of bundle isomorphisms $\varphi, \psi: D_0 \rightarrow D_1$ and every spin structure sp on ∂C_0 we have $d(\psi_*\text{sp}, \varphi_*\text{sp}) = d(\psi, \varphi)$.

Proof of (a). Since H_3 has no torsion, $v = \rho_2 \bar{v}$ for some $\bar{v} \in H_3$. Since $H_3 \cong H^1(N) \cong [N, S^1]$, the class \bar{v} is represented by an oriented 3-submanifold $P \subset N$ that is the preimage of a regular value of a map $N \rightarrow S^1$ representing \bar{v} . Denote by $V \times D^1$ a tubular neighborhood of V in N . We have that $e(\nu_0|_V) = e(\nu_0) \cap V = 0$ and that $\nu_0|_V$ is stably equivalent to the stable normal bundle of a parallelizable manifold. Hence $\nu_0|_V$ is trivial. Take a trivialization of ν_0 over $V \times D^1$, i.e. identify $\nu_0^{-1}(V \times D^1)$ and $V \times D^1 \times S^2$. Take a smooth map $\alpha: D^1 \rightarrow SO_3$ which maps a neighbourhood of the boundary to the identity and which, modulo the boundary, represents the generator of $\pi_1(SO_3) \cong \mathbb{Z}_2$. Then define

$$\varphi_1(x) := \begin{cases} \varphi(x) & x \notin \nu_0^{-1}(V \times D^1), \\ \varphi(a, t, \alpha(t)z) & x = (a, t, z) \in V \times D^1 \times S^2 = \nu_0^{-1}(V \times D^1). \end{cases}$$

By construction $d(\varphi_1, \varphi) = v$. So part (a) follows by taking $\psi := \varphi_1$.¹⁴ \square

¹⁴By (b) the equivalence class of φ_1 depends only on v not on V and on the trivialization.

Proof of (b). Carry out the construction of (a) for φ and $v := d(\psi, \varphi)$. Now (b) follows because

$$d(\psi_{*\text{sp}}, \varphi_{*\text{sp}}) = d(\varphi_{1*\text{sp}}, \varphi_{*\text{sp}}) = vd = v,$$

where

- the first equality follows because ψ is equivalent to φ_1 over the 1-skeleton;
- $a \in V$, $b \in S^2$ are any points and $d \in H^1(a \times D^1 \times b, \partial; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the relative difference class of the spin structures $\varphi_{1*\text{sp}}|_{a \times D^1 \times b}$, $\varphi_{*\text{sp}}|_{a \times D^1 \times b}$ which coincide on the boundary;
- the second equality follows by the naturality of the primary obstruction;
- the last equality is proved as follows. Take a smooth map $\alpha: D^1 \rightarrow SO_3$ which maps a neighbourhood of the boundary to the identity and which, modulo the boundary, represents the generator of $\pi_1(SO_3) \cong \mathbb{Z}_2$. Since the standard inclusion $SO_2 \rightarrow SO_3$ induces an epimorphism $\pi_1(SO_2) \rightarrow \pi_1(SO_3)$,¹⁵ we may assume that $\alpha(t) \in SO_2$ for all $t \in D^1$; i.e. that $\alpha(t)b = b$ for all $t \in D^1$. So $\varphi_v = \varphi$ over $V \times D^1 \times b$, thus $d = 1$ by definition of a spin structure. \square

Proof of the Spin Lemma 3.6.b. The result [CS11, Lemma 2.4] asserts that there is a bundle isomorphism $\varphi': \partial C_0 \rightarrow \partial C_1$. Hence using Lemma 3.7 we can modify φ' over the 1-skeleton of N to obtain a spin bundle isomorphism φ . Applying Lemma 3.7 again, and using the fact that $\pi_2(SO_3) = 0$, we obtain that φ is unique up to vertical homotopy over the 2-skeleton of N . \square

3.6 String bundle isomorphisms (Lemmas 2.5 and 3.8)

For $k = 0, 1$ let $D_k := S^7 - \text{Int } C_k$ and let st_k be the homotopy class of the restriction to D_k of the stable tangent framing on S^7 .

The following String Lemma 3.8 can be regarded as a ‘complex’ version of the Spin Lemma 3.6.

Lemma 3.8 (String Lemma). *Assume that $\kappa(f_0) = \kappa(f_1)$ and $\lambda(f_0) = \lambda(f_1)$. A bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$ is a π -isomorphism if and only if its extension $\Phi: D_0 \rightarrow D_1$ carries st_0 to st_1 .*

Remarks. (a) The ‘if’ direction of the String Lemma 3.8 is simple: If $\Phi_*\text{st}_0 = \text{st}_1$ then the stable tangent framings on C_0, C_1 coming from S^7 agree up to homotopy after identifying ∂C_0 and ∂C_1 using φ , so these framings can be glued together to give a stable tangent framing on M_φ .

(b) For a π -isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$, a stable tangent framing on M_φ whose restrictions to C_0, C_1 are induced from S^7 is not unique.

(c) The existence part of Lemma 2.5 (under weaker assumption ‘ $\lambda(f_0)(x, x) \equiv \lambda(f_1)(x, x) \pmod{2}$ for each x ’) can be proved as follows. By the regular homotopy classification of embeddings (Proposition 2.17) f_0 and f_1 are regular homotopic. (We actually only need the regular homotopy over N_0 which is easier to prove.) A regular homotopy between f_0 and f_1 extends to a regular homotopy of a tubular neighborhood of $f_0(N)$ in S^7 . By the String Lemma 3.8 the bundle isomorphism defined by this regular homotopy is a π -isomorphism.

(Below we give another proof together with the proof of the uniqueness part of Lemma 2.5.)

For the proof of Lemmas 3.8 and 2.5 we need the following definitions and lemmas.

Lemma 3.9. *For each map $\alpha: S^3 \rightarrow SO_3$ denote by $\xi(\alpha)$ the oriented 3-dimensional vector bundle over S^4*

- *whose total space is $(\mathbb{R}^3 \times D_+^4) \cup_{\phi(\alpha)} (\mathbb{R}^3 \times (-D_-^4))$, where $\phi(\alpha)(v, x) = (\alpha(v), x)$,*
- *and whose projection maps (v, x) to x .*

For a map $\alpha_1: S^3 \rightarrow SO_3$ representing $1 \in \pi_3(SO_3) = \mathbb{Z}$ we have $p_1^(\xi(\alpha_1)) = 4 \in H_0(S^4) = \mathbb{Z}$.*

¹⁵There is an alternative proof not using this fact, cf. proof of Lemma 3.11 below.

Proof. The lemma is well-known; we present the proof for completeness. We start with the identity $p_1^* := p_1^*(\xi(\alpha_1)) = \pm 4 \in H_0(S^4) = \mathbb{Z}$ [Mi56]. To determine the sign in this equation let $S\xi(\alpha_1)$ be the total space of the oriented S^2 -bundle associated to $\xi(\alpha_1)$ and $z \in H_4(S\xi(\alpha_1)) \cong \mathbb{Z}$ a generator. By [Wa66, Theorem 5] $p_1^*(S\xi(\alpha_1)) \cap z \equiv 4z^3 \pmod{24}$. This and $p_1^* = \pm 4$ imply that $p_1^*(S\xi(\alpha_1)) = 4z^2$, consequently $p_1^* = +4$. \square

For stable tangent framings st and st' on D_1 which are homotopic on the 2-skeleton of D_1 their difference

$$d(\text{st}, \text{st}') \in H^3(D_1; \pi_3(SO)) = H^3(N) = H_1,$$

is the primary obstruction to vertical homotopy between them, cf. §3.1. Here we use the zero section $N \rightarrow D^1$ to identify the cohomology groups.

A bundle isomorphism $\Phi: D_0 \rightarrow D_1$ is called *spin* if its restriction to the boundary is spin. Since the restriction induces an isomorphism $H^1(D_k; \mathbb{Z}_2) \rightarrow H^1(C_k; \mathbb{Z}_2)$, this is equivalent to carrying the spin structure on D_0 induced from S^7 to the spin structure on D_1 induced from S^7 .

Thus if $\Phi: D_0 \rightarrow D_1$ is a spin bundle isomorphism, then the difference class $d(\Phi_*\text{st}_0, \text{st}_1) \in H_1$ is well-defined by the uniqueness statements in the Spin Lemma 3.6.a,b.

Lemma 3.10. *For a spin bundle isomorphism $\Phi: D_0 \rightarrow D_1$ we have $p_{M_\varphi}^* = i_{C_0, M_\varphi} \hat{A}_0 d(\Phi_*\text{st}_0, \text{st}_1)$.*

Proof. Let

- $\delta_\varphi \in H^4(D_0 \times I, D_0 \times \partial I)$ be the primary obstruction to the extension of $\text{st}_0|_{D_0 \times 0} \cup \text{st}_1|_{D_0 \times 1}$ to a stable tangent framing of $D_0 \times I$, and
- $\gamma_\varphi \in H^4(\partial C_0 \times I, \partial)$ be the primary obstruction to the extension of $\text{st}_0|_{\partial C_0 = \partial C_0 \times 0} \cup \text{st}_1|_{\partial C_1 = \partial C_0 \times 1}$ to a stable tangent framing of $\partial C_0 \times I$.

We consider the following commutative diagram:

$$\begin{array}{ccccccc} \delta_\varphi \in H^4(D_0 \times I, D_0 \times \partial I) & \xleftarrow{\cong} & H^4(\Sigma(D_0 \times I)) & \xleftarrow{\cong} & H^3(D_0) & \xleftarrow{\cong} & H_1 & \ni d(\Phi_*\text{st}_0, \text{st}_1) \\ \downarrow \text{restriction} & & & & & \swarrow \nu' & \downarrow i_{C_0, M_\varphi} \hat{A}_0 & \\ \gamma_\varphi \in H^4(\partial C_0 \times I, \partial) & \xrightarrow{\cong} & H_3(\partial C_0 \times I) & \xrightarrow{\cong} & H_3(\partial C_0) & \xrightarrow{i_{\partial C_0, M_\varphi}} & H_3(M_\varphi) & \ni p_{M_\varphi}^* \end{array}$$

The lemma follows because by the naturality of the primary obstruction

- the image of $d(\Phi_*\text{st}_0, \text{st}_1)$ under the first line of isomorphisms is δ_φ ;
- the restriction of δ_φ is γ_φ ;
- the image of γ_φ under the second line homomorphisms is $p_{M_\varphi}^*$.

The latter statement follows because

$$M_\varphi \cong C_0 \bigcup_{\partial C_0 = \partial C_0 \times 0} \partial C_0 \times I \bigcup_{\varphi: \partial C_0 \times 1 \rightarrow \partial C_1} (-C_1)$$

and $p_{M_\varphi}^*$ is the primary obstruction to extending the spin structure on M_φ to a stable tangent framing on M_φ . (To see the latter, observe that class $p_{M_\varphi}^*$ is the primary obstruction to lifting $\bar{\tau}_{M_\varphi}: M_\varphi \rightarrow BSpin$ to γ^*EO , where $\gamma^*EO \rightarrow BSpin$ is the pullback to $BSpin$ of the universal O -bundle along the canonical map $\gamma: BSpin \rightarrow BO$.) \square

Proof of the String Lemma 3.8. By the Spin Lemma 3.6.a, it suffices to prove the result for spin φ . For each spin bundles isomorphism φ with extension $\Phi: D_0 \rightarrow D_1$ we have

$$\Phi_*\text{st}_0 = \text{st}_1 \quad \Leftrightarrow \quad d(\Phi_*\text{st}_0, \text{st}_1) = 0 \quad \Leftrightarrow \quad p_{M_\varphi}^* = 0 \quad \Leftrightarrow \quad M_\varphi \text{ is parallelizable.}$$

Here

- the first equivalence is the completeness of the obstruction;
- the second equivalence holds by Lemma 3.10 because $\kappa(f_0) = \kappa(f_1)$, $\lambda(f_0) = \lambda(f_1)$ and H_2 has no torsion, so i_{C_0, M_φ} is injective by the Agreement Lemma 3.5.b for $s = 3$;
- the last equivalence holds because $\pi_l(SO_7) = 0$ for $l = 4, 5, 6$. \square

By the Spin Lemma 3.6.b every two spin bundle isomorphisms $\Phi, \Psi: D_0 \rightarrow D_1$ are homotopic over a neighborhood of the 2-skeleton of some triangulation of N . Hence the primary (and only) obstruction,

$$d(\Phi, \Psi) \in H^3(N; \pi_3(SO_3)) = H_1,$$

to homotopy of spin bundle isomorphisms from Φ to Ψ is well-defined.

Lemma 3.11. (a) For every $v \in H_3$ and spin bundle isomorphism $\Phi: D_0 \rightarrow D_1$, there is a spin bundle isomorphism $\Phi_v: D_0 \rightarrow D_1$ such that $d(\Phi_v, \Phi) = v$.

(b) For every pair of spin bundle isomorphisms $\Phi, \Psi: D_0 \rightarrow D_1$ and for every stable tangent framing st on D_0 , we have $d(\Psi_*\text{st}, \Phi_*\text{st}) = 2d(\Psi, \Phi)$.

Proof of (a). Take an oriented circle $V \subset N$ representing v . Denote by $V \times D^3$ a tubular neighborhood of V in N . The restriction $D\nu_0|_V$ is an oriented bundle over a circle and so is trivial. Take a trivialization of $D\nu_0$ over $V \times D^3$, i.e. identify $(D\nu_0)^{-1}(V \times D^3)$ and $V \times D^3 \times B^3$. Take a smooth map $\alpha: D^3 \rightarrow SO_3$ which maps the boundary to the identity and represents (modulo the boundary) the generator $1 \in \pi_3(SO_3) \cong \mathbb{Z}$. We define Φ_1 by

$$\Phi_1(x) := \begin{cases} \Phi(x) & x \notin (D\nu_0)^{-1}(V \times D^3), \\ \Phi((a, t, \alpha(t)z)) & x = (a, t, z) \in V \times D^3 \times B^3 = (D\nu_0)^{-1}(V \times D^3). \end{cases}$$

By construction $d(\Phi_1, \Phi) = v$. □

Proof of (b). Make the construction of (a) for $v := d(\Psi, \Phi)$. Now (b) follows because

$$d(\Psi_*\text{st}, \Phi_*\text{st}) = d(\Phi_1_*\text{st}, \Phi_*\text{st}) = vd = 2v,$$

where

- first first equation follows because Ψ is equivalent over N_0 to Φ_1 ;
- $a \in V$ is any point and $d \in H^3(a \times D^3 \times 0, \partial) \cong \mathbb{Z}$ is the relative difference class of stable tangent framings $\Phi_1_*\text{st}|_{a \times D^3 \times 0}$, $\Phi_*\text{st}|_{a \times D^3 \times 0}$ which coincide on the boundary;
- the second equation follows by the naturality of the primary obstruction;
- the last equality is proved as follows. The relative difference class of tangent framings $\Phi_1_*\text{st}'|_{a \times D^3 \times 0}$, $\Phi_*\text{st}|_{a \times D^3 \times 0}$ coinciding on the boundary is 1. Since the stabilization homomorphism $\pi_3(SO_3) \rightarrow \pi_3(SO)$ is identified with multiplication by 2, we have $d = 2$. □

Lemma 3.12. If M is a closed spin 7-manifold, then p_M^* is divisible by 2.

Proof. We have $\rho_2(p_M) = w_4(M) = v_4(M) = 0$. Here

- the first equality is proved in [CS11, §3, Proof of Lemma 2.11.b];
- the second equality holds because M is spin;
- the third equality holds because $Sq^4: H^3(M; \mathbb{Z}_2) \rightarrow H^7(M; \mathbb{Z}_2)$ is trivial. □

Proof of Lemma 2.5. We use the String Lemma 3.8. Since H_1 has no 2-torsion, the uniqueness follows by Lemma 3.11.b. Let us prove the existence.

By the Spin Lemma 3.6.b there is a spin bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$. Let Φ be the extension of φ . By Lemmas 3.10 and 3.12, $i_{C_0 M_\varphi} \hat{A}_0 d(\Phi_*\text{st}_0, \text{st}_1) = p_{M_\varphi}^*$ is even. Since $\varkappa(f_0) = \varkappa(f_1)$, $\lambda(f_0) = \lambda(f_1)$ and H_2 has no torsion, i_{C_0, M_φ} is injective by the Agreement Lemma 3.5.b for $s = 3$. Since we assume that $H_3(M_\varphi, C_0) \cong H_3(C_1, \partial) \cong H_2$ is torsion free, it follows that $d(\Phi_*\text{st}_0, \text{st}_1)$ is also even; i.e., $d(\Phi_*\text{st}_0, \text{st}_1) = 2v$ for some $v \in H_1$. By Lemma 3.11.a there is a spin bundle isomorphism $\Psi: D_0 \rightarrow D_1$ such that $d(\Phi, \Psi) = -v$. Then by Lemma 3.11.b

$$d(\Psi_*\text{st}_0, \text{st}_1) = d(\Phi_*\text{st}_0, \text{st}_1) + d(\Phi_*\text{st}_0, \Psi_*\text{st}_0) = 2v - 2v = 0.$$

Then by the String Lemma 3.8 Ψ is a π -isomorphism. □

3.7 Joint Seifert classes (Lemmas 2.6, 2.9, 2.10 and 3.13)

Lemma 3.13 (Description of joint Seifert classes). *Let $\varphi: \partial C_0 \rightarrow \partial C_1$ be a bundle isomorphism and $i := i_{C_0, M_\varphi}$.¹⁶*

(a) *A class $Y \in H_5(M_\varphi)$ is a joint Seifert class if and only if $Y \cap C_k = A_k[N]$ for each $k = 0, 1$.*

(b) *Let $Y \in H_5(M_\varphi)$ be a joint Seifert class. A class $Y' \in H_5(M_\varphi)$ is a joint Seifert class if and only if*

$$Y' = Y_y := Y + i\hat{A}_0 y \quad \text{for some } y \in H_3.$$

(c) *Let $Y \in H_5(M_\varphi)$ be a joint Seifert class. Then $Y_y^2 - Y^2 = 2i\hat{A}_0 \overline{\lambda(f_0)}(y)$ for each $y \in H_3$.*

(d) *If $p \in H_4(M_\varphi)$ and $q \in H_3(C_0)$, then $p \cap_{M_\varphi} iq = A_0^{-1}(p \cap C_0) \cap_N \hat{A}_0^{-1}q$.*

Proof of (a). The ‘if’ part follows because $Y \cap_{M_\varphi} iS_{f_0}^2 = (Y \cap C_0) \cap_{C_0} S_{f_0}^2 = A_0[N] = 1$.

Let us prove the ‘only if’ part. Since $H_1(C_k) = 0$ and $H_2(C_k) \cong \mathbb{Z}$, we have $H_5(C_k, \partial) \cong \mathbb{Z}$. Since $(Y \cap C_k) \cap S_{f_k}^2 = 1$, the class $Y \cap C_k$ equals the generator $A_k[N]$ of $H_5(C_k, \partial)$. \square

Proof of (b). Look at the segment of (the Poincaré-Lefschetz dual to) the Mayer-Vietoris sequence:

$$H_5(\partial C_0) \xrightarrow{i_{\partial C_0, M_\varphi}} H_5(M_\varphi) \xrightarrow{r_0 \oplus r_1} H_5(C_0, \partial) \oplus H_5(C_1, \partial) \xrightarrow{\partial_0 - \partial_1} H_4(\partial C_0).$$

The ‘only if’ part follows because $(Y' - Y) \cap S_{f_k}^2 = 0$ and $\nu_0^!: H_3(N) \rightarrow H_5(\partial C_0)$ is an isomorphism, so $Y' - Y \in \ker(r_0 \oplus r_1) = \text{im } i_{\partial C_0, M_\varphi} = \text{im}(i\hat{A}_0)$.

The ‘if’ part follows analogously because $iS_{f_k}^2 \cap \text{im } i_{\partial C_0, M_\varphi} = 0$. \square

Proof of (c). We have

$$Y_y^2 - Y^2 = 2Y \cap_{M_\varphi} i\hat{A}_0 y = 2i((Y \cap C_0) \cap_{C_0} \hat{A}_0 y) \stackrel{(3)}{=} 2i(A_0[N] \cap_{C_0} \hat{A}_0 y) \stackrel{(4)}{=} 2i\hat{A}_0 \overline{\lambda(f_0)}(y).$$

Here (3) holds by (a) and (4) holds by Lemma 3.2.λ'. \square

Proof of (d). We have $p \cap_{M_\varphi} iq = (p \cap C_0) \cap_{C_0} q = A_0^{-1}(p \cap C_0) \cap_N \hat{A}_0^{-1}q$. Here the last equality holds by intersection Alexander duality (Lemma 3.1). \square

Proof of Lemma 2.6. Consider the following diagram:

$$H_4(M_\varphi, C_0; \mathbb{Z}_d) \xrightarrow{\partial} H_3(C_0; \mathbb{Z}_d) \xrightarrow{i} H_3(M_\varphi; \mathbb{Z}_d) \xrightarrow{j} H_3(M_\varphi, C_0; \mathbb{Z}_d) \xrightarrow{\text{ex}} H_3(C_1, \partial; \mathbb{Z}_d).$$

By Lemmas 3.13.a and 3.2.λ' we have $\text{ex } j_{C_0, M_\varphi} Y^2 = Y^2 \cap C_1 = (A_1[N])^2 = A_1 u$ with \mathbb{Z} -coefficients (these maps ex and j_{C_0, M_φ} are not to be confused with the above \mathbb{Z}_d -homomorphisms ex and j which are used elsewhere in this proof). Hence $j\rho_d Y^2 = 0$. By the Agreement Lemma 3.5.b for $s = 3$ $\partial = 0$. So i is an isomorphism onto $\ker j$. Now the existence and the uniqueness of $b_{\varphi, Y}$ follow because $\hat{A}_0 = i_{\partial C_0, C_0} \nu_0^!$ is an isomorphism.

The independence of $\beta(f_0, f_1)$ from Y for fixed φ follows by Lemma 3.13.b,c because a change of Y by Y_y leads to a change of $b_{\varphi, Y} = \hat{A}_0^{-1} i_M^{-1} \rho_d Y^2$ by $2\rho_d \overline{\lambda(f_0)}(y)$.

The independence of $\beta(f_0, f_1)$ from φ is implied by the uniqueness of Lemma 2.5, the independence of Y for fixed φ and the following Lemma 3.14 applied to $f_0 = f_1$ and $d = \text{div}(\kappa(f_0))$ (then $C_0 = C_1$ but $\varphi_0 \neq \varphi_1$ is possible). \square

Lemma 3.14. *Assume that $C_1 \supset C_0$, $H_5(C_1, C_0) = 0$, and $\varphi_k: \partial C_f \rightarrow \partial C_k$, $k = 0, 1$, are bundle isomorphisms coinciding over N_0 . Then for each joint Seifert class $Y_0 \in H_5(M_{\varphi_0})$ there is a joint Seifert class $Y_1 \in H_5(M_{\varphi_1})$ such that $i_{C_f, M_{\varphi_1}}^{-1} \rho_d Y_1^2 \subset i_{C_f, M_{\varphi_0}}^{-1} \rho_d Y_0^2 \subset H_3(C_f; \mathbb{Z}_d)$ for each d .*

¹⁶Here we do not assume that $\kappa(f_0) = \kappa(f_1)$ or $\lambda(f_0) = \lambda(f_1)$. However, we apply this lemma when $\kappa(f_0) = \kappa(f_1)$, because the existence of a joint Seifert class (Lemma 2.4) requires this assumption. If $\lambda(f_0) \neq \lambda(f_1)$, then $\lambda(f_0)(H_3) \neq \lambda(f_1)(H_3)$ is possible, (but $i_{C_0, M_\varphi} \lambda(f_0)(H_3) = i_{C_1, M_\varphi} \lambda(f_1)(H_3)$ by (b) and (c)). This is possible because i_{C_0, M_φ} or i_{C_1, M_φ} need not be injective when $\lambda(f_0) \neq \lambda(f_1)$.

Proof. Denote

$$M_k := M_{\varphi_k} \quad \text{and} \quad \overline{M}_k := C_0 \bigcup_{\varphi_k|_{N_0}: \nu_f^{-1}(N_0) \rightarrow \nu_k^{-1}(N_0)} (-C_1) \quad \text{so that} \quad M_k = \overline{M}_k \cup_{S^2 \times \partial B^5} S^2 \times B^5.$$

Since $C_1 \supset C_0$, we have $\overline{M}_1 \supset \overline{M}_0$. Consider the following diagram, where $r = r_{\overline{M}_0, \overline{M}_1}$ and the coefficients are \mathbb{Z} or \mathbb{Z}_d :

$$\begin{array}{ccccccc} H_q(M_1) & \xrightarrow{r_{M_1}} & H_q(\overline{M}_1, \partial) & \xrightarrow{r} & H_q(\overline{M}_0, \partial) & \xleftarrow{r_{M_0}} & H_q(M_0) \\ & & \uparrow j_{\partial \overline{M}_1, \overline{M}_1} & & \uparrow j_{\partial \overline{M}_0, \overline{M}_0} & & \\ & & H_q(\overline{M}_1) & \xleftarrow{i_{\overline{M}_0, \overline{M}_1}} & H_q(\overline{M}_0) & & \\ & \nwarrow i_{C_f, M_1} & \uparrow i_{\overline{M}_1} & \nearrow i_{\overline{M}_0} & & \nearrow i_{C_f, M_0} & \\ & & H_q(C_f) & & & & \end{array}$$

From the (Poincaré dual of the cohomology) exact sequence of the pair (M_1, \overline{M}_1) we obtain that r_{M_1} is an epimorphism for $q = 5$. Analogously r_{M_0} is a monomorphism for $q = 3$. By excision $H_5(\overline{M}_1, \overline{M}_0) \cong H_5(C_1, C_0) = 0$. Hence from the homology and the cohomology exact sequences of the pair $(\overline{M}_1, \overline{M}_0)$ we obtain that r is an epimorphism for $q = 5$. So we can take $Y_1 \in r_{M_1}^{-1} r^{-1} r_{M_0} Y_0 \in H_5(M_1)$. Clearly, $Y_1 \cap S_f^2 = 1$, i.e., Y_1 is a joint Seifert class for φ_1 . The required inclusion follows from $\rho_d Y_1^2 \in r_{M_1}^{-1} r^{-1} r_{M_0} \rho_d Y_0^2 \in H_3(\overline{M}_1, \partial)$ and the commutativity of the diagram because r_{M_0} is a monomorphism for $s = 3$. \square

Remark. In applications of Lemma 3.14 both subsets of $H_3(C_f; \mathbb{Z}_d)$ consist of one element $\widehat{A}_f b_{\varphi_0, Y_0} = \widehat{A}_f b_{\varphi_1, Y_1}$, where $b_{\varphi, Y}$ is defined in Lemma 2.6.

Proof of the additivity of β (Lemma 2.9). Denote $h := f \# g$. We have

$$\beta(h, f) = \beta(f, f) = 0.$$

Let us prove the second equality. (It also follows by the transitivity of β , Lemma 2.10.) The manifold $M_{\text{id} \partial C_f} = \partial(C_f \times I)$ is parallelizable. Take $Y := \partial(A_f[N] \times I) \in H_5(M_{\text{id} \partial C_f})$. Clearly, Y is a joint Seifert class for $\text{id} \partial C_f$. By Lemma 3.2. \mathcal{A} we have $A_f[N]^2 = A_f \mathcal{A}(f) \in H_3(C_f, \partial)$. Then $Y^2 \in d(\mathcal{A}(f)) H_3(M_{\text{id} \partial C_f})$. Hence $\beta(f, f) = 0$.

Let us prove the first equality. We may assume that $g(S^4) \cap C_f = \emptyset$, $\nu_f = \nu_h$ over N_0 and $C_h \supset C_f$. Since $\pi_4(V_{7,4}) = 0$ [Pa56], all embeddings $S^4 \rightarrow S^7$ are regular homotopic [Sm59]. Hence f and h are regular homotopic identically on N_0 . A regular homotopy between them extends to a regular homotopy of a tubular neighborhood of fN in S^7 , identical over $\nu_f^{-1} N_0$. The bundle isomorphism $\varphi: \partial C_f \rightarrow \partial C_h$ defined by this regular homotopy is identical over N_0 . Extend φ to a bundle isomorphism $S^7 - \text{Int } C_f \rightarrow S^7 - \text{Int } C_h$ identical over N_0 . Then by the String Lemma 3.8 φ is a π -isomorphism. Now the first equality holds by Lemma 2.6 and Lemma 3.14 for $f_0 = f$, $\varphi_0 = \text{id} \partial C_f$, Y_0 any Seifert class, $f_1 = h$, $\varphi_1 = \varphi$ and $d = d(\mathcal{A}(f))$. The assumptions of Lemma 3.14 are fulfilled because

$$H_5(C_h, C_f) \stackrel{\text{ex}}{\cong} H_5(B^4 \times D^3 - h(B^4), B^4 \times \partial D^3) \cong H_5(B^4 \times D^3 - B^4 \times 0, B^4 \times \partial D^3) = 0.$$

Here the second isomorphism holds by the 5-lemma because of the Alexander duality isomorphism $H_q(B^4 \times D^3 - h(B^4)) \cong H_{6-q}(B^4, \partial) \cong H_q(B^4 \times D^3 - B^4 \times 0)$. \square

Proof of the transitivity of β (Lemma 2.10). By Lemma 2.5 we have that there are π -isomorphisms $\varphi_{01}: \partial C_0 \rightarrow \partial C_1$ and $\varphi_{21}: \partial C_2 \rightarrow \partial C_1$. Let

$$V := M_{\varphi_{01}} \times [-1, 0] \bigcup_{-C_1 \times 0 = -C_1 \times 0} M_{\varphi_{21}} \times [0, 1].$$

Denote $\varphi_{kl} := \varphi_{lk}^{-1}$. Observe that $M_{\varphi_{kl}} = -M_{\varphi_{lk}}$. Then $\partial V = M_{\varphi_{10}} \sqcup M_{\varphi_{21}} \sqcup M_{\varphi_{02}}$.

In this paragraph $k \in \{10, 21\}$. Take any $x \in H_3$. By Lemmas 2.4 and 3.5.c there are joint Seifert classes $Y_{4,k} \in H_5(M_{\varphi_k})$ and $Y_{3,k} \in H_4(M_{\varphi_k})$, $Y_{3,k}$ for x . Denote $I_{10} = [-1, 0]$ and $I_{21} = [0, 1]$. Applying the Mayer-Vietoris sequences for V we see that there is a class $\overline{Y}_q \in H_{q+2}(V, \partial)$ such that

$$\overline{Y}_q \cap (M_{\varphi_k} \times I_k) = Y_{q,k} \times I_k \in H_{q+2}(M_{\varphi_k} \times I_k, \partial) \quad \text{for each } k \in \{10, 21\}, \quad q \in \{3, 4\}.$$

Then for each $q \in \{3, 4\}$ the class $Y_{q,20} := \partial \overline{Y}_q \cap M_{\varphi_{20}} \in H_{q+1}(M_{\varphi_{20}})$ is a joint Seifert class for f_2 , f_0 and φ_{20} , where $Y_{3,20}$ corresponds to x . So the triple $(V, \overline{Y}_4, \overline{Y}_3)$ is a cobordism between

$$(M_{\varphi_{20}}, Y_{4,20}, Y_{3,20}) \quad \text{and} \quad (M_{\varphi_{10}}, Y_{4,10}, Y_{3,10}) \sqcup (M_{\varphi_{21}}, Y_{4,21}, Y_{4,21}).$$

Since $Y_{4,rl}^2 \cap Y_{3,rl}$ is a characteristic number of such triples,

$$Y_{3,20} \cap_{M_{\varphi_{20}}} Y_{4,20}^2 = Y_{3,10} \cap_{M_{\varphi_{10}}} Y_{4,10}^2 + Y_{3,21} \cap_{M_{\varphi_{10}}} Y_{4,21}^2 \in \mathbb{Z}.$$

Denote $d := \text{div}(\kappa(f_0))$. By Lemma 2.6 there are $b_{rl} := \hat{A}_l^{-1} i_{C_l, M_{\varphi_{rl}}}^{-1} \rho_d Y_{4,rl}^2 \in H_1(N; \mathbb{Z}_d)$. Then by Lemma 3.13.d $Y_{3,rl} \cap_{M_{\varphi_{rl}}} Y_{4,rl}^2 = x \cap_N b_{rl} \in \mathbb{Z}_d$. Hence $x \cap_N (b_{20} - b_{10} - b_{21}) = 0 \in \mathbb{Z}_d$. Since this holds for each $x \in H_3$ and H_1 is torsion free, $b_{20} = b_{10} + b_{21} \in H_1(N; \mathbb{Z}_d)$. By the String Lemma 3.8 the composition $\varphi_{20} := \varphi_{01}^{-1} \varphi_{21}$ is a π -isomorphism. Hence taking the quotients modulo $2\lambda(f_0)(H_3)$ of both sides we obtain $\beta(f_2, f_0) = \beta(f_1, f_0) + \beta(f_2, f_1)$. \square

3.8 Calculations of the β -invariant (Lemmas 2.7 and 2.13 for β)

Remark. The calculation of β (Lemma 2.7.b) follows by Remark 2.12 and the parametric additivity (Lemma 2.13). Lemma 2.13 for β is proved using the particular case Lemma 2.7.a of Lemma 2.7.b. For this reason, as well as for applications, it is convenient to state Lemmas 2.7.a,b separately.

Lemma 3.15 (proved below in §3.8). *Let $f_0, f_1: S^1 \times S^3 \rightarrow S^7$ be embeddings such that $\lambda(f_0) = \lambda(f_1) = 0$. Then* ¹⁷

$$i_{C_0, M_\varphi} \hat{A}_0 \beta(f_0, f_1) = Y^2 - \frac{1}{4} p_1^*(M_\varphi) \in H_3(M_\varphi)$$

for any bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$ such that M_φ is spin and any joint Seifert class $Y \in H_5(M_\varphi)$.

Lemma 3.16 (proved below in §3.8). *Let $f_0, f_1: S^1 \times S^3 \rightarrow S^7$ be embeddings and suppose that $\varphi, \varphi': \partial C_0 \rightarrow \partial C_1$ are bundle isomorphisms such that $\varphi = \varphi'$ over $S^1 \times D_-^3$, and over $S^1 \times D_+^3$ the bundle isomorphism φ is obtained from φ' by twisting with the $+1 \in \pi_3(SO_3) = \mathbb{Z}$ (i.e. for the extension $\Phi: D_0 \rightarrow D_1$, $N = S^1 \times S^3$ and $V := S^1 \times 1_3$ we have $\varphi' = \Phi_1|_{\partial C_0}$ in the notation of the proof of Lemma 3.11.a.). Then the triple*

$$(M_{\varphi'}, Y_4', Y_3') \quad \text{is cobordant to} \quad (M_\varphi, Y_4, Y_3) \sqcup (\mathbb{CP}^3 \times S^1, [\mathbb{CP}^2 \times S^1], [\mathbb{CP}^2 \times 1_1])$$

for some joint Seifert classes $Y_q \in H_{q+1}(M_\varphi)$ and $Y_q' \in H_{q+1}(M_{\varphi'})$, $q = 3, 4$, where Y_3 and Y_3' are for $1_1 \times S^3$.

Lemmas 3.16, 4.6 and [Sk08, Cobordism Lemma] are analogous.

¹⁷There is a unique $x \in H_3(M_\varphi)$ such that $4x = p_1^*(M_\varphi)$. This follows by the proof of the lemma (or because $p_1(M_\varphi)$ is divisible by 4 by Lemma 3.12 and $H_3(M_\varphi) \cong \mathbb{Z}$; one proves the latter using the agreement of Seifert classes, Lemma 3.5.b, and the exact sequence of pair (M_φ, C_0)).

Proof of Lemma 3.15. By Lemma 2.5 there is a π -isomorphism $\partial C_0 \rightarrow \partial C_1$. If $\varphi : \partial C_0 \rightarrow \partial C_1$ is a π -isomorphism, then M_φ is parallelizable, so $p_1(M_\varphi) = 0$, hence the required equality holds by definition of $\beta(f_0, f_1)$.

The proof of Lemma 2.5 shows that *each spin bundle isomorphism $\varphi : \partial C_0 \rightarrow \partial C_1$ can be modified by a sequence of twistings with $\pm 1 \in \pi_3(SO_3) = \mathbb{Z}$ to obtain a π -isomorphism φ'* . Hence it suffices to prove that *if φ, φ' are spin bundle isomorphisms and φ is obtained from φ' by twisting with $\pm 1 \in \pi_3(SO_3) = \mathbb{Z}$, then $i_{C_0, M_\varphi}^{-1}(Y^2 - \frac{1}{4}p_1^*(M_\varphi)) = i_{C_0, M_{\varphi'}}^{-1}((Y')^2 - \frac{1}{4}p_1^*(M_{\varphi'}))$ for some joint Seifert classes $Y \in H_5(M_\varphi)$ and $Y' \in H_5(M_{\varphi'})$* .

Let us prove this assertion. Take any joint Seifert classes $Y_3 \in H_4(M_\varphi)$ and $Y'_3 \in H_4(M_{\varphi'})$ for $1_1 \times S^3$. The map $(i_{C_0, M_\varphi} \hat{A}_0)^{-1} : H_3(M_\varphi) \rightarrow H_1(S^1 \times S^3) = \mathbb{Z}$ is an isomorphism coinciding with $x \mapsto x \cap_{M_\varphi} Y_3$. Analogous assertion holds with φ, Y, Y_3 replaced by φ', Y', Y'_3 . Now the assertion for ‘+1-modification’ follows because by Lemma 3.16

$$(Y'_4)^2 \cap_{M_{\varphi'}} Y'_3 - Y_4^2 \cap_{M_\varphi} Y_3 = [\mathbb{C}P^2 \times S^1]^2 \cap_{\mathbb{C}P^3 \times S^1} [\mathbb{C}P^2 \times 1_1] = [\mathbb{C}P^1 \times S^1] \cap_{\mathbb{C}P^3 \times S^1} [\mathbb{C}P^2 \times 1_1] = 1$$

$$\text{and } p_1^*(M_{\varphi'}) \cap_{M_{\varphi'}} Y'_3 - p_1^*(M_\varphi) \cap_{M_\varphi} Y_3 = p_1^*(\mathbb{C}P^3 \times S^1) \cap_{\mathbb{C}P^3 \times S^1} [\mathbb{C}P^2 \times 1_1] = 4.$$

The assertion for ‘−1-modification’ is analogous. \square

Proof of Lemma 3.16. Take a map $\alpha : S^3 \rightarrow SO_3$ representing $+1 \in \pi_3(SO_3)$ and such that $\alpha|_{D_-^3} = \text{id } S^2$. Identify ∂C_0 with $S^2 \times S^1 \times S^3$ by any bundle isomorphism. Identify ∂C_1 with $\partial C_0 = S^2 \times S^1 \times S^3$ by φ . Define a self-diffeomorphism

$$\begin{aligned} \bar{\alpha} \text{ of } (S^2 \times S^1 \times S^3 \times I) - \left(S^2 \times S^1 \times \text{Int } D_+^3 \times \left[\frac{1}{3}, \frac{2}{3} \right] \right) \\ \text{by } \bar{\alpha}(x) := \begin{cases} (\alpha(b)a, z, b, t) & x = (a, z, b, t) \in S^2 \times S^1 \times D_+^3 \times [\frac{2}{3}, 1] \\ x & \text{otherwise.} \end{cases} \end{aligned}$$

Let

$$V := (C_0 \times I) \bigcup_{\bar{\alpha}} (C_1 \times I) \quad \text{and} \quad \Sigma := S^2 \times D^3 \times \left[\frac{1}{3}, \frac{2}{3} \right].$$

Then V is a cobordism between $M_{\varphi'}$ and $M_\varphi \sqcup E_\alpha \times S^1$, where

$$\hat{\alpha} : \partial \Sigma \rightarrow \partial \Sigma \quad \text{maps } (a, b, t) \text{ to } \begin{cases} (a, b, t) & t < 2 \\ (\alpha(b)a, b, 2) & t = 2 \end{cases}$$

$$\text{and } E_\alpha := \Sigma \cup_{\hat{\alpha}} \Sigma \stackrel{(1)}{\cong} \frac{S^2 \times D^4}{\{(a, b) \sim (\alpha(b)a, R(b))\}_{(a, b) \in S^2 \times D_+^3}} \stackrel{(2)}{\cong} \mathbb{C}P^3.$$

Here $R : D_+^3 \rightarrow D_-^3$ is the reflection with respect to $\partial D_+^3 = \partial D_-^3$. The diffeomorphism (2) is well-known and is proved using a retraction to the dual $(\mathbb{C}P^1)^* \subset \mathbb{C}P^3$ of the complement to a tubular neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^3$.¹⁸

By the Agreement Lemma 3.5.a $\varphi \partial A_0 = \partial A_1$ and $\varphi' \partial A_0 = \partial A_1$. Hence using the (Poincaré dual of the cohomology) Mayer-Vietoris sequence for V we see that for each $q \in \{3, 4\}$ there are

$$\bar{Y}_q \in H_{q+2}(V, \partial) \quad \text{such that} \quad \bar{Y}_4 \cap (C_k \times I) = A_k[S^1 \times S^3] \times I \in H_6(C_k \times I, \partial)$$

$$\text{and } \bar{Y}_3 \cap (C_k \times I) = A_k[1_1 \times S^3] \times I \in H_5(C_k \times I, \partial) \quad \text{for each } k = 0, 1.$$

¹⁸Alternatively, (2) follows because E_α is an S^2 -bundle over S^4 with characteristic map representing $+1 \in \pi_3(SO_3)$.

Then $Y_q := \partial \overline{Y}_q \cap M_\varphi$ and $Y'_q := \partial \overline{Y}'_q \cap M_{\varphi'}$ are joint Seifert classes; Y_4 is for $[S^1 \times S^3]$ and Y_3 is for $[1_1 \times S^3]$. Hence $(V, \overline{Y}_4, \overline{Y}_3)$ is a cobordism between

$$(M_{\varphi'}, Y'_4, Y'_3) \quad \text{and} \quad (M_\varphi, Y_4, Y_3) \sqcup (S^1 \times E_\alpha, \partial \overline{Y}_4 \cap (S^1 \times E_\alpha), \partial \overline{Y}_3 \cap (S^1 \times E_\alpha)).$$

We have $\partial \overline{Y}_4 \cap (S^1 \times E_\alpha) = [S^1] \otimes y_4$ for a certain $y_4 \in H_4(E_\alpha)$. Since

$$\overline{Y}_4 \cap (C_0 \times I) = A_0[S^1 \times S^3] \times I, \quad \text{we have} \quad y_4 \cap \Sigma = \left[* \times D_-^3 \times \left[\frac{1}{3}, \frac{2}{3} \right] \right] \in H_4(\Sigma, \partial).$$

Hence under (1) y_4 goes to a class whose intersection with $[S^2 \times 0]$ in the quotient manifold is $+1$. Therefore under the composition of (1) and (2) y_4 goes to a class whose intersection with $[\mathbb{C}P^1]$ in $\mathbb{C}P^3$ is $+1$, i.e. to $[\mathbb{C}P^2]$.

We have $\partial \overline{Y}_3 \cap (S^1 \times E_\alpha) = * \otimes y_3$ for a certain $y_3 \in H_4(E_\alpha)$. Analogously to the above under the composition of (1) and (2) y_3 goes to $[\mathbb{C}P^2]$. \square

Take a map $\alpha: S^3 \rightarrow \pi_3(V_{4,2})$ representing the element $(0, b)$. In this subsection we abbreviate the subscript τ_k to k , e.g. $\nu_\alpha = \nu_{\tau_\alpha}$.

Proof of the calculation of β (Lemma 2.7.a). Take a normal vector field e_1 on $S^3 \subset S^7$ tangent to $D^4 \supset S^3$ and pointing outside D^4 . Take the standard framing $S^3 \times D^3 \rightarrow S^7$ of the orthogonal complement to e_1 in the normal bundle of $S^3 \subset S^7$. Take a normal vector field e_2 on $S^3 \subset S^7$ orthogonal to e_1 and representing an element $b \in \pi_3(S^2)$ w.r.t. the standard framing. Since the ‘action’ map $\pi_3(SO_3) \rightarrow \pi_3(S^2)$ is an isomorphism, e_2 can be completed to a framing e_2, e_3, e_4 of the orthogonal complement to e_1 , and this framing is unique up to homotopy. Recall that $\text{im } \tau_\alpha$ is formed by ends of ε -length vectors normal to $S^3 \subset S^7$ in the subbundle spanned by e_1 and e_2 of the normal bundle to S^3 in S^7 (for some small ε).

Recall that $\nu_\alpha = \nu_{\tau_\alpha}$ is the normal bundle of τ_α . Take a section ξ_α of ν_α in the 2-plane subbundle, spanned by e_1 and e_2 , of the normal bundle to $S^3 \subset S^7$, so that for each point $(x, y) \in S^1 \times S^3$ the vector $\xi_\alpha(x, y)$ looks into the 2-disk bounded by $\tau_\alpha(S^1 \times y)$ but not outside. Then ξ_α, e_3, e_4 is a framing of ν_α .

For each $x \in S^3$ the circle $\xi_\alpha(S^1 \times x) \subset S^3 \times x$ bounds a 2-disk in $x \times D^4$. The union of such 2-disks along $x \in S^3$ is a compact 5-manifold $W_\alpha \cong D^2 \times S^3$ with boundary $\xi_\alpha(S^1 \times S^3)$. Choose the orientation on W_α so that $\partial W_\alpha = \xi_\alpha(S^1 \times S^3)$. Since $A_\alpha^{-1} = \nu_\alpha \partial$ and $\nu_\alpha \xi_\alpha = \text{id } N$, the manifold W_α represents the relative homology class $[W_\alpha] = A_\alpha[S^1 \times S^3] \in H_5(C_\alpha, \partial)$.

Let

$$D_\alpha^4 := (1 - \varepsilon)D^4 \subset S^7 \quad \text{and} \quad S_1^3 := 1_1 \times S^3.$$

We have $\partial D_\alpha^4 = \xi_\alpha(S_1^3)$. Hence $[D_\alpha^4] = A_\alpha[S_1^3] \in H_4(C_\alpha, \partial)$ for a certain orientation on D_α^4 .

Make analogous construction for α replaced by the constant map α_0 . We obtain the standard embedding $\tau_0 = \tau_{\alpha_0}$, a section ξ_0 , a framing,

$$[W_0] = A_0[S^1 \times S^3] \in H_5(C_0, \partial) \quad \text{and} \quad [D_0^4] = A_0[S_1^3] \in H_4(C_0, \partial).$$

Take a bundle isomorphism $\varphi: \partial C_0 \rightarrow \partial C_\alpha$ defined by the constructed framings. We may assume that $f_0 = f_\alpha$ over a neighborhood of $S^1 \times 1_3$. Hence φ carries the spin structure sp_α on ∂C_α coming from S^7 to the spin structure sp_0 on ∂C_0 coming from S^7 . Therefore M_φ is spin. Since $\varphi \xi_0 = \xi_\alpha$, we have

$$\varphi \partial W_0 = \varphi \xi_0(S^1 \times S^3) = \xi_\alpha(S^1 \times S^3) = \partial W_\alpha \quad \text{and} \quad \varphi \partial D_0^4 = \varphi \xi_0(S_1^3) = \xi_\alpha(S_1^3) = \partial D_\alpha^4.$$

Hence $W_0 \cup_\partial (-W_\alpha)$ and $\Sigma^4 := D_0^4 \cup_\partial (-D_\alpha^4)$ with their natural orientations represent joint Seifert classes $Y_4 \in H_5(M_\varphi)$ and $[\Sigma^4] \in H_4(M_\varphi)$, where $[\Sigma^4]$ corresponds to $[S_1^3]$. We have

$$W_0 \cap D_0^4 = W_\alpha \cap D_\alpha^4 = \emptyset \quad \Rightarrow \quad (W_0 \cup_\partial (-W_\alpha)) \cap \Sigma^4 = \emptyset \quad \Rightarrow \quad Y_4 \cap_{M_\varphi} [\Sigma^4] = 0 \quad \Rightarrow \quad Y_4^2 \cap_{M_\varphi} [\Sigma^4] = 0,$$

$$\text{and } p_1^*(M_\varphi) \cap_{M_\varphi} [\Sigma^4] = p_1^*(\tau_{M_\varphi}|_{\Sigma^4}) = p_1^*(\Sigma^4) + p_1^*(\mu) = -4b,$$

where

- μ is the normal bundle of Σ^4 in M_φ ;
- the last equality follows because S^4 is stably parallelizable, the map $p_1: \pi_3(SO_3) \rightarrow \mathbb{Z}$ is multiplication by 4 [DNF12], and μ corresponds to the preimage of $-b$ under the ‘action’ map $\pi_3(SO_3) \rightarrow \pi_3(S^2)$ because the obstruction to existence of a non-zero section of μ is $-b$.

Let us prove the latter statement. Take the normal vector field e_2^0 on $S^3 \subset S^7$ orthogonal to e_1 and representing the constant map $S^3 \rightarrow S^2$ w.r.t. the standard framing. The corresponding section of the normal bundle of $\partial D_0^4 \subset S^7$ is tangent to ∂C_0 . That section is mapped under $d\varphi$ to the section of the normal bundle of $\partial D_\alpha^4 \subset S^7$ corresponding to e_2 (here $\varphi: \partial C_0 \rightarrow \partial C_\alpha$ is thought of as a diffeomorphism rather than a bundle isomorphism). Clearly, e_2^0 extends to a non-zero section of the normal bundle of $D_0^4 \subset S^7$. The obstruction to extension of e_2 to a non-zero section of the normal bundle of $D_\alpha^4 \subset S^7$ is b . Since μ is a bundle over $D_0^4 \cup_\partial (-D_\alpha^4)$, the obstruction to the existence of a non-zero section of μ is $-b$.

Thus $\beta(\tau_0, \tau_\alpha) \cap_{S^1 \times S^3} [S_1^3] = (Y_4^2 - \frac{1}{4}p_1^*(M_\varphi)) \cap_{M_\varphi} [\Sigma^4] = b$ by Lemmas 3.15 and 3.13.d. Hence $\beta(\tau_0, \tau_\alpha) = b[S^1 \times 1_3]$. \square

Proof of the parametric additivity of β (Lemma 2.13). We use the notation from §3.3. Analogously to Lemma 2.5 one proves that *there is a π -isomorphism $\psi: \partial C_\alpha \rightarrow \partial C_0$ such that $\psi(\partial C_\alpha \cap D_\pm^7) \subset (\partial C_0 \cap D_\pm^7)$.*

For an embedding g denote $C_{g\pm} := C_g \cap D_\pm^7$. Observe that

$$C_{f-} = C_{\alpha-} = C_{0-} \cong C_{0+}, \quad C_{h+} = C_{f+} \quad \text{and} \quad C_{h-} = C_{\alpha+}.$$

Then $\text{id } C_{f+} \cup R$ gives diffeomorphisms

$$C_h \cong C_{f+} \bigcup_{C_{f+} \cap \partial D_+^7 = C_{\alpha+} \cap \partial D_+^7} C_{\alpha+} \quad \text{and} \quad \partial C_h \cong \partial C_f \cap D_+^7 \bigcup_{\partial C_f \cap \partial D_+^7 = \partial C_\alpha \cap \partial D_+^7} \partial C_\alpha \cap D_+^7$$

(here $C_{\alpha+}$ comes with the positive orientation because R preserves the orientation). Identify ∂C_h with the right-hand side of the latter equality. Let $\varphi: \partial C_h \rightarrow \partial C_f$ be $\text{id}(\partial C_f \cap D_+^7) \cup \psi|_{\partial C_\alpha \cap D_+^7}$. Then by the String Lemma 3.8 φ is a π -isomorphism. (This is proved by considering the restrictions of stable tangent framings coming from S^7 to $\partial C_f \cap D_+^7$, $\partial C_\alpha \cap D_+^7$ and $\partial C_0 \cap D_+^7$.)

Let

$$V := M_{\text{id } \partial C_f} \times [-1, 0] \bigcup_{R_3} (-M_\psi) \times [0, 1],$$

where $R_3: (C_{f-} \cup (-C_{f-})) \times 0 \rightarrow ((-C_{\tau-}) \cup C_{\tau_0,-}) \times 0$ is the union of two copies of the reflection with respect to the hyperplane $x_3 = 0$. Then $\partial V = M_\varphi \sqcup (-M_\psi) \sqcup (-M_{\text{id } \partial C_f})$. Now the proof is completed analogously to the second paragraph of the proof of β -transitivity (Lemma 2.10) using the calculation of β (Lemma 2.7.a). \square

Remark. The following result (not used in this paper) is proved analogously to Lemma 3.15. *If $\lambda(f_0) = \lambda(f_1)$, $\varkappa(f_0) = \varkappa(f_1)$, $\varphi: \partial C_0 \rightarrow \partial C_1$ is a bundle isomorphism such that M_φ is spin and $Y \in H_5(M_\varphi)$ is a joint Seifert class, then $\beta(f_0, f_1) = [\hat{A}_0^{-1} i_{C_0, M_\varphi}^{-1} \rho_{\text{div}(\varkappa(f_0))}(Y^2 - \frac{1}{4}p_1^*(M_\varphi))] \in K_{\lambda(f_0), \varkappa(f_0)}$.*

4 Proof of the MK Isotopy Classification Theorem 2.8

4.1 The obstruction $\eta(\varphi, Y)$ and its properties

Recall that some notation was introduced in §§1.2, 1.4, 2.1 and 3.1.

Denote $d_0 := \text{div } \kappa(f_0)$. In what follows, a statement involving k holds for both $k = 0, 1$.

A joint Seifert class $Y \in H_5(M_\varphi)$ is called a d -class for an integer d if $\rho_d Y^2 = 0$. (If the group $H_3(M_\varphi)$ is free, this is equivalent to $Y^2 \in dH_3(M_\varphi)$.)

Lemma 4.1. *If $\lambda(f_0) = \lambda(f_1)$, $\kappa(f_0) = \kappa(f_1)$, $\beta(f_0, f_1) = 0$ and $\varphi: \partial C_0 \rightarrow \partial C_1$ is a π -isomorphism, then there is a d_0 -class for φ .*

Lemma 4.1 follows from Lemmas 2.4 and 3.13.bc together with the definition of $\beta(f_0, f_1)$.

For a compact 8-manifold W we consider the intersection products

$$\cap_\partial: H_4(W) \times H_4(W, \partial) \rightarrow \mathbb{Z} \quad \text{and} \quad \cap_{\partial\partial}: H_6(W, \partial) \times H_6(W, \partial) \rightarrow H_4(W, \partial).$$

As before, for the corresponding squares $H_6(W, \partial) \rightarrow H_4(W, \partial)$ we do not put any subscript.

The following lemma is well-known.

Lemma 4.2. *If a compact manifold has a spin structure, then it has a (stable) normal spin structure, i.e. a stable normal framing over the 2-skeleton of some triangulation.*

Definition of $\eta(\varphi, Y)$ for a π -isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$ and a d -class $Y \in H_5(M_\varphi)$. Since $\varphi: \partial C_0 \rightarrow \partial C_1$ is a π -isomorphism, M_φ is spin. Take any normal spin structure on M given by Lemma 4.2. Since M_φ is simply-connected, a normal spin structure on M_φ is unique. Since $\Omega_7^{\text{Spin}}(\mathbb{C}P^\infty) = 0$ [KS91, Lemma 6.1] there is a compact 8-manifold W with a normal spin structure and $z \in H_6(W, \partial)$ such that $\partial W \stackrel{\text{spin}}{=} M_\varphi$ and $\partial z = Y$. Consider the following fragment of the exact sequence of the pair $(W, \partial W)$:

$$H_4(\partial W; \mathbb{Z}_d) \xrightarrow{i_W} H_4(W; \mathbb{Z}_d) \xrightarrow{j_W} H_4(W, \partial; \mathbb{Z}_d) \xrightarrow{\partial_W} H_3(\partial W; \mathbb{Z}_d).$$

Since $\partial_W \rho_d z^2 = \rho_d Y^2 = 0$, there is a class $\overline{z^2} \in H_4(W; \mathbb{Z}_d)$ such that $j_W \overline{z^2} = \rho_d z^2$. Define

$$\eta(\varphi, Y) = \eta(f_0, f_1, d, \varphi, Y) := \rho_{\widehat{d}}(\overline{z^2} \cap_\partial (z^2 - p_W^*)) \in \mathbb{Z}_{\widehat{d}}, \quad \text{where} \quad \widehat{d} := \gcd(d, 24).$$

Proof that $\eta(\varphi, Y)$ is well-defined, i.e. independent of the choice of W, z and $\overline{z^2}$. The proof is analogous to [CS11, 2.3 and footnote (q)]. For the independence from $\overline{z^2}$ instead of [CS11, Lemma 2.7] we use $\partial_W p_W^* = p_{M_\varphi}^* = 0$. For the independence from W, z instead of the uniqueness of $\partial_W z$ of [CS11, Lemma 2.6] we use that $\partial_W z = Y$ is fixed. \square

Lemma 4.3 (proved in §4.3). *Let $\varphi: \partial C_0 \rightarrow \partial C_1$ be a π -isomorphism and $Y \in H_5(M_\varphi)$ a d_0 -class.*

(a) (Divisibility of η by 2) *The residue $\eta(\varphi, Y) \in \mathbb{Z}_{\widehat{d}_0}$ is even.*

(b) (Change of η) *There is an embedding $g: S^4 \rightarrow S^7$, a π -isomorphism $\varphi': \partial C_0 \rightarrow \partial C_{f_1 \# g}$ and a d_0 -class $Y' \in H_5(M_{\varphi'})$ such that $\eta(\varphi', Y', f_0, f_1 \# g, d_0) = \eta(\varphi, Y, f_0, f_1, d_0) + 2$.*

(c) (Change of φ) *For every π -isomorphism $\varphi': \partial C_0 \rightarrow \partial C_1$ there is a d_0 -class $Y' \in H_5(M_{\varphi'})$ such that $\eta(\varphi', Y') = \eta(\varphi, Y)$.*

Note that Lemma 4.3.a is trivial for \widehat{d}_0 odd.

Other properties of η which are not used in this paper will be discussed in [CSII].

4.2 Proof of Theorem 2.8 using Theorem 4.5 and Lemmas 4.3 and 4.4

In the definition of $\eta(\varphi, Y)$ in §4.1 instead of C_0, C_1, φ, Y we can take any pair of simply-connected parallelizable 7-manifolds M_0, M_1 , a diffeomorphism $\varphi: \partial M_0 \rightarrow \partial M_1$ such that the manifold $M := M_0 \cup_\varphi (-M_1)$ is parallelizable and any class $Y \in H_6(M)$ such that $\rho_d Y^2 = 0$. Denote by $\eta(\varphi, Y) = \eta(M_0, M_1, d, \varphi, Y) \in \mathbb{Z}_{\widehat{d}}$ the obtained residue. Also, in this situation for d even we can define

$$\eta'(\varphi) = \eta'(M_0, M_1, d, \varphi) := \rho_2(\overline{z^2} \cap_\partial z^2) \in \mathbb{Z}_2.$$

The proof that $\eta'(\varphi)$ is independent of the choice of W, z and $\overline{z^2}$ is analogous to the case of $\eta(\varphi, Y)$.

Lemma 4.4. Assume that d_0 is even and $\varphi: \partial C_0 \rightarrow \partial C_1$ is a π -isomorphism.

(a) (proved in §4.4) The residue $\eta'(\varphi)$ is well-defined, i.e. is independent of the choice of Y .

(b) (Change of η' ; proved in §4.3) There is a π -isomorphism $\varphi': \partial C_0 \rightarrow \partial C_1$ such that $\eta'(\varphi') = \eta'(\varphi) + 1$.

Theorem 4.5 (Almost Diffeomorphism Theorem; proved in §4.5). *Let*

- M_0, M_1 be oriented simply-connected 7-manifolds whose homology groups are free abelian and such that $H_5(M_k, \partial) \cong \mathbb{Z}$;

- $\varphi: \partial M_0 \rightarrow \partial M_1$ be a diffeomorphism such that $M := M_0 \cup_\varphi (-M_1)$ is a parallelizable oriented manifold for which $H_2(M), H_3(M)$ are free abelian and $\tilde{j}_k := j_{M_k, M}: H_4(M) \rightarrow H_4(M, M_k)$, $k = 0, 1$, are epimorphisms having the same kernel;

- $Y \in H_5(M)$ be a class such that $Y \cap M_k$ is a generator $\alpha_k \in H_5(M_k, \partial)$, $\text{div}(Y^2) = \text{div}(\alpha_0^2) = \text{div}(\alpha_1^2) =: d$ and there is a class $Q \in H_4(\partial M_0)$ such that $i_M Q \cap_M Y^2 = d$.

For some homotopy 7-sphere Σ there is an orientation-preserving diffeomorphism $\bar{\varphi}: M_0 \rightarrow M_1 \# \Sigma$ extending φ and such that $\bar{\varphi}\alpha_0 = \alpha_1 \# 0$ if ¹⁹

$$\eta(\varphi, Y) = 0 \quad \text{and, for } d \text{ even, } \eta'(\varphi) = 0.$$

Proof of Theorem 2.8 using the Almost Diffeomorphism Theorem 4.5. Since $\lambda(f_0) = \lambda(f_1)$ and $\varkappa(f_0) = \varkappa(f_1)$, by Lemma 2.5 there is a π -isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$. Since $\beta(f_0, f_1) = 0$, by Lemma 4.1 there is a d_0 -class $Y \in H_5(M_\varphi)$.

By the divisibility of $\eta(\varphi, Y)$ by 2 (Lemma 4.3.a) and by a change of η (Lemma 4.3.b) we can change f_1 (by connected sum with a knot), φ and Y , and assume that $\eta(\varphi, Y) = 0$.

In this paragraph assume that d_0 is even. Then by a change of η' (Lemma 4.4.b) we obtain a new π -isomorphism φ such that $\eta'(\varphi) = 0$. By a change of φ (Lemma 4.3.c) we obtain a new d_0 -class Y such that $\eta(\varphi, Y) = 0$.

Since $7 = 4 + 3$, by general position C_k are simply-connected. The groups $H_3(C_k) \xrightarrow{\hat{A}_k} H_1$ and $H_4(C_k, \partial) \xrightarrow{A_k} H_3$ are free abelian. Hence by Lefschetz duality $H_2(C_k)$ is free abelian. So all homology groups of C_k are free abelian. Since $\varkappa(f_0) = \varkappa(f_1)$ and $\lambda(f_0) = \lambda(f_1)$,

- by the exact sequence of the pair (M_φ, C_k) and the Agreement Lemma 3.5.b for $q = 2$ the group $H_2(M_\varphi)$ is free abelian;

- by the Agreement Lemma 3.5.b for $q = 3$ the map $j_k: H_4(M_\varphi) \rightarrow H_4(M_\varphi, C_k)$ is onto.

Since $\hat{A}_k: H_2 \rightarrow H_4(C_k)$ is an isomorphism and $i_{C_1, M_\varphi} \hat{A}_1 = i_{\partial C_1, M_\varphi} \varphi \nu_0^! = i_{C_0, M_\varphi} \hat{A}_0$, we have $\text{im } i_{C_0, M_\varphi} = \text{im } i_{C_1, M_\varphi}$. Hence $\ker j_{C_0, M_\varphi} = \ker j_{C_1, M_\varphi}$. Then from the exact sequence of the pair (M_φ, C_0) we obtain that $H_3(M_\varphi)$ and $H_2(M_\varphi)$ are free abelian.

By Alexander duality, $\alpha_k := A_k[N]$ is a generator of $H_5(C_k, \partial)$. We have $d_0 = \text{div}(\varkappa(f_0)) = \text{div}(\alpha_k^2)$. By Lemmas 3.13.a and 3.5.c there is a class $Y \in H_5(M)$ such that $Y \cap C_k = \alpha_k$. Since Y^2 is divisible by d_0 and $A_0 \varkappa(f_0) = A_0(Y \cap C_0)^2$, we have $\text{div}(Y^2) = d_0$.

Take $Q' \in H_2$ such that $Q' \cap_N \varkappa(f_0) = d_0$. Let $Q := \nu_0^! Q' \in H_4(\partial C_0)$. Then

$$i_{\partial C_0, M_\varphi} Q \cap_{M_\varphi} Y^2 = i_{C_0, M_\varphi} \hat{A}_0 Q' \cap_{M_\varphi} Y^2 = \hat{A}_0 Q' \cap_{C_0} (Y \cap C_0)^2 = Q' \cap_N \varkappa(f_0) = d_0.$$

Hence by the Almost Diffeomorphism Theorem 4.5 for $M_0 = C_0$, $M_1 = C_1$ and $M = M_\varphi$ there is an orientation preserving diffeomorphism $\bar{\varphi}: C_0 \rightarrow C_1 \# \Sigma$ extending the bundle isomorphism φ . The bundle isomorphism φ also extends to an orientation-preserving diffeomorphism $S^7 - \text{Int } C_0 \rightarrow S^7 - \text{Int } C_1$. Therefore $S^7 \cong S^7 \# \Sigma \cong \Sigma$. Hence f_0 and f_1 are isotopic by Lemma 1.5. \square

¹⁹We conjecture that the ‘only if’ statement is also true. For its proof we can take $W := M_0 \times I$, $z := \alpha_0 \times I$. Presumably z has to be modified using Q so that $\partial z = Y$. Then we find \bar{z}^2 so that $\bar{z}^2 \cap z^2 = 0 \in \mathbb{Z}_d$. This and $p_W = 0$ implies that $\eta(\varphi, Y) = 0$ and, for d even, $\eta'(\varphi) = 0$.

4.3 Proofs of Lemmas 4.3 and 4.4.b

Proof of Lemma 4.3.a. Take any pair (W, z) from the definition of $\eta(\varphi, Y)$ and take some map $Z: W \rightarrow \mathbb{C}P^\infty$ corresponding to $z \in H_6(W, \partial) \cong H^2(W) \cong [W, \mathbb{C}P^\infty]$. By spin surgery of Z relative to ∂W we may assume that Z is 3-connected. The residue $\overline{z^2} \cap_\partial (z^2 - p_W^*)$ does not change throughout this surgery because it is ‘spin $\mathbb{C}P^\infty$ -characteristic residue modulo d relative to the boundary’. Since Z is 3-connected, by the Hurewicz Theorem for the mapping cylinder of Z we have $H_3(W) = \pi_3(W) = \pi_3(\mathbb{C}P^\infty) = 0$. Hence $\text{Tors } H_4(W) \cong \text{Tors } H_3(W) = 0$. So there is a class $\widehat{z^2} \in H_4(W)$ such that $\rho_{d_0} \widehat{z^2} = \overline{z^2}$. Then

$$\overline{z^2} \cap_\partial (z^2 - p_W^*) = \rho_{d_0}(\widehat{z^2} \cap_\partial z^2 - \widehat{z^2} \cap_\partial p_W^*) = \rho_{d_0}(\widehat{z^2} \cap_\partial z^2 - \widehat{z^2} \cap_\partial p_W^*).$$

The latter residue is divisible by 2 by [CS11, Lemma 2.11]. \square

Lemma 4.3.b is proved analogously to [CS11, §3, the second equality of Addendum 1.3].

For the proofs of Lemmas 4.3.c and 4.4.b we need the following result.

Lemma 4.6 (proved below in §4.3). *Assume that π -isomorphisms $\varphi', \varphi: \partial C_0 \rightarrow \partial C_1$ coincide over N_0 and that over $\text{Cl}(N - N_0)$ they differ by the generator of $\pi_4(SO_3) \cong \mathbb{Z}_2$. Then for each integer d and d -class $Y \in H_5(M_\varphi)$ there is a d -class $Y' \in H_5(M_{\varphi'})$ such that the pair*

$$(M_{\varphi'}, Y') \text{ is cobordant to } (M_\varphi, Y) \sqcup (S^2 \tilde{\times} S^5, A),$$

where $S^2 \tilde{\times} S^5$ is the total space of the non-trivial S^2 -bundle over S^5 (i.e. the bundle corresponding to the non-trivial element of $\pi_4(SO_3) \cong \mathbb{Z}_2$) and $A \in H_5(S^2 \tilde{\times} S^5) \cong \mathbb{Z}$ is a generator.

Proof of Lemma 4.3.c. By Lemma 2.5 we may assume that $\varphi' = \varphi$ over N_0 . We may also assume that over $\text{Cl}(N - N_0)$ isomorphism φ' obtained from φ by twisting with $d(\varphi', \varphi) \in \pi_4(SO_3) \cong \mathbb{Z}_2$. If $d(\varphi', \varphi) = 0$, then we may assume that $\varphi' = \varphi$ and take $Y' = Y$. If $d(\varphi', \varphi) \neq 0$, then by Lemma 4.6 and a calculation in [CS11, Proof of the Framing Theorem 2.9] $\eta(\varphi', Y') = \eta(\varphi, Y)$. \square

Proof of Lemma 4.4.b. We do not assume Lemma 4.4.a. and so we write $\eta'(\varphi, Y)$ instead of $\eta'(\varphi)$ and prove the lemma in the following form.

For each π -isomorphism $\varphi: \partial C_0 \rightarrow \partial C_1$, even integer d and d -class $Y \in H_5(M_\varphi)$ there is a π -isomorphism $\varphi': \partial C_0 \rightarrow \partial C_1$ and a d -class $Y' \in H_5(M_{\varphi'})$ such that $\eta'(\varphi', Y') = \eta'(\varphi, Y) + 1$.

Take a bundle isomorphism $\varphi': \partial C_0 \rightarrow \partial C_1$ coinciding with φ over N_0 and over $\text{Cl}(N - N_0)$ obtained from φ by twisting with the non-trivial element of $\pi_4(SO_3) \cong \mathbb{Z}_2$. Then by the String Lemma 3.8 φ' is a π -isomorphism. By Lemma 4.6 and a calculation in [CS11, Proof of the Framing Theorem 2.9] $\eta'(\varphi', Y') = \eta'(\varphi, Y) + 1$. \square

Proof of Lemma 4.6. Take a smooth map $\alpha: S^4 \rightarrow SO_3$ representing the non-trivial element of $\pi_4(SO_3) \cong \mathbb{Z}_2$ and such that $\alpha|_{D^4} = \text{id } S^2$. For $k = 0, 1$ identify

$$\nu_k^{-1} \text{Cl}(N - N_0) \times \left[\frac{1}{3}, \frac{2}{3} \right] \quad \text{with} \quad \Sigma_k := S^2 \times D^4 \times \left[\frac{1}{3}, \frac{2}{3} \right] \quad (\text{so } \Sigma_0 = \Sigma_1).$$

Let $U_k := \partial C_k \times I - \text{Int } \Sigma_k$. Define

$$\overline{\alpha}: U_0 \rightarrow U_1 \quad \text{by} \quad \overline{\alpha}(s, t) := \begin{cases} (\varphi(s), t) & s \in \nu_0^{-1}(N - N_0), t \in [\frac{2}{3}, 1] \\ (\varphi'(s), t) & \text{otherwise} \end{cases}$$

$$\text{and} \quad V := C_0 \times I \bigcup_{\overline{\alpha}} C_1 \times I.$$

Hence V is a cobordism between $M_{\varphi'}$ and $M_{\varphi} \sqcup E_{\alpha}$, where

$$\widehat{\alpha}: \partial\Sigma_0 \rightarrow \partial\Sigma_1 \quad \text{maps } (a, b, t) \text{ to } \begin{cases} (a, b, t) & t < \frac{2}{3} \\ (\alpha(b)a, b, \frac{2}{3}) & t = \frac{2}{3} \end{cases}$$

$$\text{and } E_{\alpha} := \Sigma_0 \cup_{\widehat{\alpha}} \Sigma_1 \stackrel{(1)}{\cong} \frac{S^2 \times D^5}{\{(s, b) \sim (\alpha(b)s, R(b))\}_{(s, b) \in S^2 \times D_+^4}} \stackrel{(2)}{\cong} S^2 \tilde{\times} S^5.$$

Here $R: D_+^4 \rightarrow D_-^4$ is the reflection with respect to $0 \times \mathbb{R}^4$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} H^q(V, C_0 \times I) & \xrightarrow{\text{ex} \cong} & H^q(C_1 \times I, U_1) & & \\ \downarrow i & & \downarrow & \swarrow i_{C_1 \times I} & \\ H^q(M_{\varphi}, C_0) & \xrightarrow{\text{ex} \cong} & H^q(C_1, \partial) & \xrightarrow{i_{C_1, C_1 \times I} \cong} & H^q(C_1 \times I, \partial C_1 \times I) \end{array}$$

We have

$$H^q(\partial C_1 \times I, U_1) \stackrel{\text{ex}}{\cong} H^q(\Sigma_1, \partial) \cong H_{7-q}(\Sigma_1) = 0 \quad \text{for } q = 1, 2, 3, 4.$$

Hence from the exact sequence of the triple $U_1 \subset \partial C_1 \times I \subset C_1 \times I$ we see that $i_{C_1 \times I}$ is injective for $q = 2, 3, 4, 5$. Hence i is an isomorphism for $q = 2, 3, 4, 5$. Look at the inclusion-induced mapping of the exact sequences of pairs (M_{φ}, C_0) and $(V, C_0 \times I)$. By the 5-lemma we see that the inclusion $M_{\varphi} \rightarrow V$ induces an isomorphism in $H^q(\cdot)$ for $q = 2, 4$. Or, in Poincaré dual form, $r_{M_{\varphi}} \partial_V: H_q(V, \partial) \rightarrow H_{q-1}(M_{\varphi})$ is an isomorphism for $q = 4, 6$. The same holds for φ replaced by φ' .

Let $\overline{Y} := (r_{M_{\varphi}} \partial_V)^{-1} Y \in H_6(V, \partial)$. Then by Lemma 3.13.a

$$\overline{Y} \cap (C_k \times I) = A_k[N] \times I \in H_6(C_k \times I, \partial) \quad \text{for each } k = 0, 1.$$

So by Lemma 3.13.a $Y' := r_{M_{\varphi'}} \partial_V \overline{Y} \in H_5(M_{\varphi'})$ is a joint Seifert class. We have the equation $\rho_d(Y')^2 = \rho_d r_{M_{\varphi'}} \partial_V (r_{M_{\varphi}} \partial_V)^{-1} Y^2 = 0$, i.e. Y' is a d -class. Let $Y_{\alpha} := \partial \overline{Y} \cap E_{\alpha} \in H_5(E_{\alpha})$. Since

$$\overline{Y} \cap (C_0 \times I) = A_0[N] \times I, \quad \text{we have } Y_{\alpha} \cap \Sigma_1 = \left[* \times D^4 \times \left[\frac{1}{3}, \frac{2}{3} \right] \right] \in H_5(\Sigma_1, \partial).$$

Hence under (1) Y_{α} goes to a class whose intersection with $[S^2 \times 0]$ is +1. Therefore under the composition of (1) and (2) Y_{α} goes to a class whose intersection with the fiber S^2 is +1, i.e. to A .

Therefore (V, \overline{Y}) is the required cobordism. \square

4.4 Proof of Lemma 4.4.a

Definition of M_f and $Y_{f,y}$. Identify C_f and $C_f \times 0$. Denote

$$M_f := \partial(C_f \times I) = M_{\text{id} \partial C_f} \quad \text{and, for } y \in H_3, \quad Y_{f,y} := \partial(A_f[N] \times I) + i_{C_f, M_f} \widehat{A}_f y \in H_5(M_f).$$

Lemma 4.7 (Description of d -classes for M_f). *A class $Y \in H_5(M_f)$ is a d -class if and only if $Y = Y_{f,y}$ for some $y \in \ker(2\rho_{\text{div}(\kappa(f))} \overline{\lambda}(f))$.*

This follows by Lemma 3.13.b,c.

Lemma 4.8 (proved below in §4.4). *For each $y \in H_3$ there is a spin null-bordism (W, z) of $(M_f, Y_{f,y})$ such that p_W^* is even.²⁰*

²⁰We cannot take $W = C_f \times I$ because $\partial H_5(C_f \times I, \partial) \not\ni Y_{f,y}$. So we note that the following equality holds $\partial(A_f[N] \times I + \widehat{A}_f y \times I) = Y_{f,y} + \widehat{A}_f y \times 1 \neq Y_{f,y}$ and ‘surger out’ $\widehat{A}_f y \times 1$ shifted into the interior.

Proof of Lemma 4.4.a. Before we prove that $\eta'(\varphi)$ is independent of Y we denote it by $\eta'(\varphi, Y)$. Take any pair of d_0 -classes $Y', Y'' \in H_5(M_\varphi)$. We have

$$\eta'(\varphi, Y') - \eta'(\varphi, Y'') \stackrel{(1)}{=} \eta'(\text{id } \partial C_{f_0}, Y) \stackrel{(2)}{=} \eta'(\text{id } \partial C_{f_0}, Y_{f_0, y}) \stackrel{(3)}{=} 0 \in \mathbb{Z}_2,$$

where

- equality (2) holds for some $y \in \ker(2\rho_{\text{div}(\kappa(f_0))} \overline{\lambda(f_0)})$ by the description of d -classes for M_f (Lemma 4.7);
- equality (3) holds by Lemmas 4.3.a and 4.8;
- equality (1) holds for some d_0 -class $Y \in H_5(M_{f_0})$ by the following result.

Let $f_0, f_1, f_2: N \rightarrow S^7$ be embeddings, $\varphi_{01}: \partial C_0 \rightarrow \partial C_1$ and $\varphi_{12}: \partial C_1 \rightarrow \partial C_2$ π -isomorphisms, $Y_{01} \in H_5(M_{\varphi_{01}})$ and $Y_{12} \in H_5(M_{\varphi_{12}})$ d -classes. Then $\varphi_{02} := \varphi_{12}\varphi_{01}$ is a π -isomorphism and there is a d -class $Y_{02} \in H_5(M_{\varphi_{02}})$ such that $\eta'(\varphi_{02}, Y_{02}) = \eta'(\varphi_{01}, Y_{01}) + \eta'(\varphi_{12}, Y_{12})$.

This result is proved analogously to [CS11, Lemma 2.10], cf. [Sk08', §2, Additivity Lemma] (the property that Y_{02} is a d -class is achieved analogously to the proof of Lemma 4.6). \square

Definition of a simplifying 6-bordism V and maps $v = v_0, v_1, v_2, v_3$. A simplifying 6-bordism for f and an oriented 3-submanifold $P \subset N$ is a 6-manifold $V \subset C_f$ with boundary $\partial V = \nu_f^{-1}P \sqcup v(S^2 \times S^3)$ for some embedding $v = v_0: S^2 \times S^3 \rightarrow \text{Int } C_f$ such that $V \cap \partial C_f = \nu_f^{-1}P$ and $v(S^2 \times 1_3)$ is homologous to S_f^2 in C_f . (Then $[\text{im } v] = \widehat{A}_f[P] \in H_5(C_f)$.)

E.g. for $N = S^1 \times S^3$ and $P = 1_1 \times S^3$ we can take a simplifying 6-bordism $S^2 \times S^3 \times I \cong V \subset C_f$.

Let $v_1: S^2 \times S^3 \times D^1 \rightarrow \text{Int } C_f$ be an embedding such that $v_1|_{S^2 \times S^3 \times 1} = v$, $\text{im } v_1 \cap V = \text{im } v$ and $v_1(c \times D^1)$ is tangent to V for each $c \in S^2 \times S^3$.

Extend v_1 to an orientation-preserving embedding $v_2: S^2 \times S^3 \times D^2 \rightarrow \text{Int } C_f = \text{Int } C_f \times \frac{1}{2}$ transversal to V and such that $\text{im } v_2 \cap V = \text{im } v$.

Extend v_2 to an orientation-preserving embedding $v_3: S^2 \times S^3 \times D^3 \rightarrow \text{Int}(C_f \times I)$.

Lemma 4.9. *For each oriented 3-submanifold $P \subset N$ there is a simplifying 6-bordism.*

Proof. Equip $\nu_f^{-1}P$ with the spin structure induced from C_f . (This spin structure is compatible with the orientation of $\nu_f^{-1}P$.) Since C_f is simply connected, we can perform spin surgeries on 1-spheres in the 5-manifold $\nu_f^{-1}P$ to obtain a spin bordism between the inclusion $\nu_f^{-1}P \rightarrow C_f$ and a map $\mu: X \rightarrow C_f$ of some closed simply connected 5-manifold X . Since the induced map $i_{C_f}: H_2(\nu_f^{-1}P) \rightarrow H_2(C_f) \cong \mathbb{Z}$ is surjective, $\mu: H_2(X) \rightarrow H_2(C_f)$ is surjective. By Smale's classification of simply connected spin 5-manifolds [Sm62, Theorem A] (see also [Cr11, Theorem 4.1]), there is a closed simply connected spin 5-manifold X' with $H_2(X') = \ker \mu$. Choose any isomorphism $H_2(X) \rightarrow \ker \mu \oplus \mathbb{Z}$. Then by Barden's classification of simply connected 5-manifolds [Ba65, Theorem 2.2] (see also [Cr11, Theorem 5.1]), we may identify X with $(S^2 \times S^3) \# X'$ so that $\mu H_2(X') = \{0\} \subset H_2(C_f)$. Also by Smale's classification [Sm62, Theorem 1.1], X' is spin diffeomorphic to the boundary of a handlebody obtained by attaching 3-handles to D^6 . So the co-cores of these handles give framed embeddings of 2-spheres such that spin surgery on these 2-spheres gives S^5 . Applying this to 2-spheres in X' and using $\mu H_2(X') = 0 \in H_2(C_f)$ we obtain a spin bordism over C_f , $g: V \rightarrow C_f$, between μ and a map $S^2 \times S^3 \rightarrow C_f$ inducing an isomorphism on H_2 . Then

- V is a spin 6-manifold obtained from $\nu_f^{-1}P \times I$ by attaching 2-handles $D^2 \times D^4$ and 3-handles $D^3 \times D^3$ to $\nu_f^{-1}P \times 1$;
- $\partial V \stackrel{\text{spin}}{=} \nu_f^{-1}P \times 0 \sqcup S^2 \times S^3$;
- $g|_{\nu_f^{-1}P \times 0}$ is the identity and $g|_{S^2 \times S^3}$ induces an isomorphism on H_2 .

Now the lemma follows by (the second part of) the following Semioper Embedding Theorem 4.10.b for $\partial_+ V := \nu_f^{-1}P$ and $\partial_- V := S^2 \times S^3$. \square

Theorem 4.10 (Semiproper Embedding). *Let V and X be compact v - and x -manifolds such that $\partial V = \partial_+ V \sqcup \partial_- V$. Then every map $g: V \rightarrow X$ such that $g|_{\partial_+ V}$ is an embedding into ∂X is homotopic rel $\partial_+ V$ to an embedding $V \rightarrow X$, provided either*

- (a) $x < 2v$ and $(V, \partial_- V)$ is $(2v - x - 1)$ -connected, or
- (b) $x = 7 = v + 1$ and $(V, \partial_- V)$ is 2-connected, X and V are spin and the spin structure on $\partial_+ V$ is the g -preimage of the spin structure on ∂X .²¹

Proof. First assume that $(V, \partial_- V)$ is $(2v - x - 1)$ -connected. Then there is a handle decomposition of V relative to $\partial_+ V$ without handles of index more than $v - (2v - x - 1) - 1 = x - v$. In particular, V is a regular neighborhood (in itself) of an $(x - v)$ -polyhedron.

Use induction on the number of handles. The base case is $V = \partial_+ V \times I$ (when there are no handles). Then define an embedding $V \rightarrow X$ as $\partial_+ V \times I \xrightarrow{g|_{\partial_+ V} \times \text{id}_I} \partial X \times I \xrightarrow{i_X} X$, where i_X is the collar inclusion.

Let us prove the inductive step for (a). We may assume that $V' := V \cup D^k \times D^{v-k}$, $k \leq x - v$ and $g: V' \rightarrow X$ is a map such that $g|_{\partial_+ V}$ is an embedding into ∂X and $g|_V$ is an embedding. Since $x < 2v$, we have $x \geq 2(x - v) + 1 \geq 2k + 1$. Since V is a regular neighborhood (in itself) of an $(x - v)$ -polyhedron, we may assume that $g|_{D^k \times 0}$ is an embedding and $g(D^k \times 0) \cap g(V) = g(\partial D^k \times 0)$. Since $k \leq x - v$, we have $\pi_{k-1}(V_{x-k, v-k}) = 0$. Hence the normal $(v - k)$ -framing of $g(\partial D^k \times 0)$ in $g(\partial_- V)$ extends to a normal $v - k$ framing of $g(D^k)$ in X . Thus $g|_{D^k \times 0}$ extends to an embedding $D^k \times D^{v-k} \rightarrow X$ whose image intersects $g(V)$ at $g(\partial D^k \times D^{v-k})$. This extension defines an embedding $V' \rightarrow X$ extending $g|_V$ and homotopic to g .

Now we prove (b). By hypothesis, there is a handle decomposition of V relative to $\partial_+ V$ without handles of index more than $6 - 2 - 1 = 3$. The proof is the same as the proof of (a) above except that $x \geq 2k + 1$ is verified directly and $k > x - v = 1$ is possible. The required extension of the $(v - k)$ -framing exists

- because for $k = 3$ we have $\pi_{k-1}(V_{x-k, v-k}) = \pi_2(SO_4) = 0$;
- because for $k = 2$ (in spite of $\pi_{k-1}(V_{x-k, v-k}) = \pi_1(SO_5) \neq 0$) V' is spin and the spin structure on V is the g -preimage of the spin structure on X . \square

Proof of Lemma 4.8. Take any $y \in H_3$. Since $H_3 \cong H^1(N) \cong [N, S^1]$, the class y is represented by an oriented 3-submanifold $P \subset N$ that is the preimage of a regular value of a map $N \rightarrow S^1$ representing y ; orientations on N and S^1 give an orientation on the preimage. Take a simplifying 6-bordism $V \subset C_f$ given by Lemma 4.9. Take the corresponding maps v, v_1, v_2, v_3 . Let

$$W_- := (C_f \times I) - \text{Int im } v_3 \quad \text{and} \quad W := W_- \cup_{v_3|_{S^2 \times S^3 \times S^2}} (S^2 \times D^4 \times S^2).$$

(The manifold W may be called the result of an S^2 -parametric surgery along v_3 .)

Denote

$$t := v_3(S^2 \times 0 \times S^2) \quad \text{and} \quad \Delta := 1_2 \times D^4 \times 1_1.$$

Identify $S^2 \times D^4 \times S^2$ with $t \times \Delta$.

Consider the cohomology exact sequence of the pair (W, W_-) in the following Poincaré dual form:

$$(*) \quad \begin{array}{ccccccc} H_6(t \times \Delta) & \longrightarrow & H_6(W, \partial) & \xrightarrow{r_{W_-}} & H_6(W_-, \partial) & \longrightarrow & H_5(t \times \Delta) \\ \uparrow \cong & & & & & & \uparrow \cong \\ PD_{\text{dex}} & & & & & & PD_{\text{dex}} \\ H^2(W, W_-) & & & & & & H^3(W, W_-) \end{array}$$

Since $H_5(t \times \Delta) = 0$, the map r_{W_-} is an epimorphism. Take any

$$Z \in r_{W_-}^{-1}(A_f[N] \times I \cap W_-) \subset H_6(W, \partial).$$

²¹We only use (b). However, (a) illustrates the idea of the proof of (b), and (a) is hopefully of independent interest.

Denote

$$\widehat{V} := V \cup (S^2 \times D^4 \times 1) \subset W \quad \text{and} \quad z := Z + [\widehat{V}] \in H_6(W, \partial).$$

Objects constructed above depend upon y, f and the choices in the construction. We do not indicate this in their notation.

Since $H_6(t \times \Delta) = 0$, the spin structure on W_- coming from $S^7 \times I$ extends to W . Clearly, $\partial W \underset{\text{spin}}{=} \partial(C_f \times I) \underset{\text{spin}}{=} M_f$ (for the ‘boundary’ spin structure on M_f coming from $C_f \times I$). Since

$$\partial_W Z = \partial_{C_f \times I}(A_f[N] \times I) = Y_{f,0} \quad \text{and} \quad \partial_W[\widehat{V}] = [\nu_f^{-1}P \times \frac{1}{2}] = i_{M_f}\widehat{A}_f y, \quad \text{we have} \quad \partial_W z = Y_{f,y}.$$

Consider the first line of diagram (*) with subscripts 6,5 changed to 4,3, respectively. Since $p_W^* \cap W_- = p_{W_-}^* = 0$, by exactness $p_W^* = n[t]$ for some $n \in \mathbb{Z}$. Denote

$$W'_+ := (S^7 - \text{Int im } v_2) \cup_{v_2|_{S^2 \times S^3 \times S^1}} S^2 \times D^4 \times S^1.$$

Then

$$n = n[t] \cap_{t \times \Delta} [\Delta] = (p_W^* \cap t \times \Delta) \cap_{t \times \Delta} [\Delta] \stackrel{(3)}{=} (p_{W'_+}^* \cap S^2 \times D^4 \times S^1) \cap_{S^2 \times D^4 \times S^1} [\Delta] \equiv 0 \pmod{2}.$$

Here

- the homology classes $[t]$ and $[\Delta]$ are taken in the space indicated under ‘ \cap ’ (so $[\Delta]$ has different meanings in different parts of the formula);
- the equality (3) holds because $r_{S^2 \times D^4 \times S^1}: H_4(t \times \Delta, \partial) \rightarrow H_3(S^2 \times D^4 \times S^1, \partial)$ is an isomorphism;
- the congruence holds because $H_5(S^2 \times D^4 \times S^1) = 0$, so the spin structure on $S^7 - \text{Int im } v_2$ coming from S^7 extends to W'_+ , hence by Lemma 3.12 $p_{W'_+}^*$ is even. \square

4.5 Proof of Theorem 4.5 using Lemmas 4.12 and 4.13

Definition of an elementary pair. Suppose that U, V_0 and V_1 are abelian groups and that $\cap_{01}: V_0 \times V_1 \rightarrow \mathbb{Z}$ a unimodular pairing. (Then V_k has to be free abelian.) An *elementary* pair is a pair $v_k: U \rightarrow V_k$, $k = 0, 1$, of monomorphisms such that $v_0 U \cap_{01} v_1 U = 0$ and $v_k U$ is a half-rank direct summand in V_k for each $k = 0, 1$. (Then $\text{rk } V_k$ has to be even.)

The following theorem is an easy corollary of a theorem of Kreck. In it and in §4.7 we consider the intersection product

$$\cap_{01}: H_4(W, M_0) \times H_4(W, M_1) \rightarrow \mathbb{Z}.$$

Theorem 4.11 (Modified surgery theorem). *For $l \geq 2$ let*

- $M_0, M_1 \subset \mathbb{R}^{8l}$ be compact $(4l-1)$ -manifolds with common boundary;
- $p: B \rightarrow BO$ be a fibration such that $\pi_1(B) = 0$ and $\pi_i(p) = 0$ for each $i \geq 2l$;
- $\overline{S\nu}_k: M_k \rightarrow B$, $k = 0, 1$, be $(2l-1)$ -connected maps coinciding on the boundary and such that $p\overline{S\nu}_k$ is the classifying map of the normal bundle of M_k .

A diffeomorphism $M_0 \rightarrow M_1$ commuting with $\overline{S\nu}_k$ and identical on ∂M_0 exists if there is

- a compact $4l$ -manifold W such that $\partial W = M_0 \cup (-M_1)$,
- a $2l$ -connected²² map $\overline{S\nu}: W \rightarrow B$ extending $\overline{S\nu}_0 \cup \overline{S\nu}_1$,
- a subgroup $U \subset \ker \overline{S\nu} \subset H_{2l}(W)$ such that the pair $j_{M_k, W}|_U$, $k = 0, 1$, is elementary.

²²We conjecture that this assumption is superfluous. We can always assume that $\overline{S\nu}$ is $2l$ -connected, by making B -surgery below the middle dimension and so changing $\overline{S\nu}$ relative to the boundary. However, it is not easy to see that the surgery preserves the existence of U as below.

Proof. For an elementary pair $v_k: U \rightarrow V_k$, $k = 0, 1$, the quotient $v_0U \times V_1/v_1U \rightarrow \mathbb{Z}$ of \cap_{01} is unimodular. So by [CS11, the Kreck Theorem 4.1], cf. [Kr99, Theorem 4], $\overline{S\nu}$ is bordant (relative to the boundary) to an h -cobordism. Hence the theorem holds by the relative h -cobordism theorem [Mi65]. \square

Definitions of i_W, j_W, ∂_W , convenient manifold and pre-elementary class. Let W be a compact 8-manifold.

Denote by i_W, j_W, ∂_W the homomorphisms from the exact sequence of the pair $(W, \partial W)$.

The manifold W is called *convenient* if $H_3(\partial W)$ is free abelian, $H_5(W, \partial) = H_3(W) = 0$ and ∂W is parallelizable.

A class $z \in H_6(W, \partial)$ is called *pre-elementary* if there is a homomorphism $s: H_4(W, \partial) \rightarrow H_4(W)$ such that

- (1) $H_4(W) = \text{im } i_W \oplus \text{im } s$,
- (2) $su \cap_W sv = su \cap_\partial v$ for each $u, v \in H_4(W, \partial)$, and
- (3) $\sigma(W) = sp_W^* \cap_\partial p_W^* = sz^2 \cap_\partial z^2 = sz^2 \cap_\partial p_W^* = 0$.

Lemma 4.12 (Pre-elementary class; proved in §4.6). *Let W be a convenient 8-manifold and $z \in H_6(W, \partial)$ a class such that for $d := \text{div}(\partial_W z^2)$ and some $\overline{z^2} \in H_6(W, \mathbb{Z}_d)$*

$$j_W \overline{z^2} = \rho_d z^2, \quad \overline{z^2} \cap_\partial (z^2 - p_W^*) \equiv 0 \pmod{\widehat{d}} \quad \text{and, if } d \text{ is even, } \overline{z^2} \cap_\partial z^2 \equiv 0 \pmod{2}.$$

Then there is a spin 8-manifold W' such that $\partial W'$ is a homotopy 7-sphere and $z \# 0 \in H_6(W \# W', \partial)$ is pre-elementary.

Lemma 4.13 (Elementary pair; proved in §4.7). *Let W be a convenient 8-manifold such that*

() $\partial W = M_0 \cup_{\partial M_0 = \partial M_1} (-M_1)$ for some 7-manifolds M_0, M_1 without torsion in their homology and having a common boundary, and*

*(**) $j_{M_k, \partial W}: H_4(\partial W) \rightarrow H_4(\partial W, M_k)$, $k = 0, 1$, are epimorphisms having the same kernel.*

Let $z \in H_6(W, \partial)$ be a pre-elementary class for which there is a class $q \in \ker j_{M_0, \partial W}$ such that $q \cap_{\partial W} \partial_W z^2 = \text{div}(\partial_W z^2)$.

Then there is a subgroup $U \subset H_4(W)$ such that $U \cap_\partial z^2 = U \cap_\partial p_W^ = 0$ and the pair of homomorphisms $j_{M_k, W}|_U$, $k = 0, 1$, is elementary.*

We remark that the proofs of the Elementary pair Lemma 4.13 modulo Lemma 4.15 (found in §4.7) and of the Pre-elementary class Lemma 4.12 modulo Lemma 4.14 (found in §4.6) are similar to [CS11, Proof of Bordism Theorem 4.3 and of Lemma 4.5]. However, these proofs are different in details from those in [CS11], even when $H_1 = 0$.

Proof of the Almost Diffeomorphism Theorem 4.5 modulo Lemmas 4.12 and 4.13. Take the spin structure on M corresponding to a tangent framing on M . Take any normal spin structure on M given by Lemma 4.2. Since $\Omega_7^{\text{Spin}}(\mathbb{C}P^\infty) = 0$ [KS91, Lemma 6.1] there is a compact 8-manifold W with a normal spin structure and $z \in H_6(W, \partial)$ such that $\partial W \stackrel{\text{spin}}{=} M$ and $\partial z = Y$.

Recall that $B\text{Spin} = BO\langle 4 \rangle$ is the (unique up to homotopy) 3-connected space for which there exists a fibration $\gamma: B\text{Spin} \rightarrow BO$ inducing an isomorphism on π_i for each $i \geq 4$. Let $B := B\text{Spin} \times \mathbb{C}P^\infty$, $p := \gamma \text{pr}_2$ and $\overline{S\nu}: W \rightarrow B$ be the map corresponding to the given normal spin structure on W and to $z \in H_6(W, \partial) \cong H^2(W) \cong [W, \mathbb{C}P^\infty]$.

For each $k = 0, 1$ since M_k is torsion free, $H_2(M_k) \cong H_5(M_k, \partial) \cong \mathbb{Z}$. Then the homomorphism $\overline{S\nu}|_{M_k}: H_2(M_k) \rightarrow H_2(\mathbb{C}P^\infty)$ is an isomorphism. This and the fact that $\pi_1(M_k) = 0$ imply that the map $\overline{S\nu}|_{M_k}$ is 3-connected.

Performing B -surgery below the middle dimension we can change $\overline{S\nu}$ relative to the boundary and assume that $\overline{S\nu}$ is 4-connected [Kr99, Proposition 4]. Then

$$H_5(W, \partial) \cong H^3(W) \cong H^3(B) = 0, \quad H_3(W) \cong H_3(B) = 0 \quad \text{and} \quad H_2(W) \cong H_2(B) \cong \mathbb{Z}.$$

From Poincaré-Lefschetz duality it follows that $\text{im } j_W$ is a direct summand in $H_4(W, \partial)$. The manifold W is now convenient. By the Pre-elementary class Lemma 4.12 we can change W to obtain a new manifold, again denoted W , with $\partial W = M_0 \# \Sigma$ and $z \in H_6(W, \partial)$ elementary.

Let $q := i_M Q$. Take a subgroup U given by the Elementary pair Lemma 4.13. Recall that there is an isomorphism $H_4(B) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ mapping $\overline{S\nu}(x)$ to $(x \cap_{\partial} z^2, x \cap_{\partial} p_W^*)$ for each $x \in H_4(W)$. Then $U \subset \ker \overline{S\nu}$. Apply the Modified Surgery Theorem 4.11 for $l = 2$ and $\overline{S\nu}_k := \overline{S\nu}|_{M_k}$. The obtained diffeomorphism commutes with $\overline{S\nu}_k$ and so is orientation-preserving. \square

4.6 Proof of the Pre-elementary class Lemma 4.12

We first construct a homomorphism s satisfying (1) from the definition of a pre-elementary class (and some additional properties). Then we show how to achieve (2) keeping (1), and finally we show how to achieve (3) keeping (1) and (2).

Lemma 4.14. *Let V, V' be free abelian groups, $\cdot : V \times V' \rightarrow \mathbb{Z}$ a unimodular form, $j : V \rightarrow V'$ a homomorphism whose image is a direct summand and $(\text{im } j)^\perp = \ker j$.*

A homomorphism $s : V' \rightarrow V$ is called 1-homomorphism if

$$V = \ker j \oplus \text{im } s, \quad jsj = j \quad \text{and} \quad sjs = s.$$

A homomorphism $s : V' \rightarrow V$ is called a 2-homomorphism if

$$V = \ker j \oplus \text{im } s \quad \text{and} \quad x \cdot jsv = x \cdot v \quad \text{for each } x \in \text{im } s, v \in V'.$$

(a) *There is a 1-homomorphism.*

(b) *For a 1-homomorphism $t : V' \rightarrow V$ define a homomorphism $t^* : \text{im } t \rightarrow V$ by the property*

$$t^*x \cdot u = x \cdot jtu \quad \text{for each } u \in V'.$$

*Then t^*t is a 2-homomorphism.*

(c) *If s is a 2-homomorphism, then*

(c1) $jsj = j$;

(c2) $V' = \text{im } j \oplus \ker s$;

(c3) $sjs = s$;

(c4) t is a 1-homomorphism for each homomorphism $t : V' \rightarrow \text{im } s$ such that $tj = sj$.

Proof of (a). Since V and V' are free abelian, there is a subgroup $T \subset V$ such that $V = \ker j \oplus T$. Then $j|_T$ is injective and $j(T) = \text{im } j$. Since $j(T) = \text{im } j$ is a direct summand in V' , the inverse of the abbreviation $j : T \rightarrow j(T)$ extends to an epimorphism $t : V' \rightarrow T$. We have $tjt = t$ and $jtj = j$. \square

Proof of (b). Denote $s := t^*t$. Take any $x \in \text{im } t$. Since $jtj = j$, we have $t^*x \cdot jtu = x \cdot jtjtu = x \cdot jtu$ for each $u \in V'$. Hence $t^*x - x \perp \text{im}(jt)$. Since $V = \ker j \oplus \text{im } t$, we have $\text{im}(jt) = \text{im } j$. Then $t^*x - x \in \ker j$, i.e. $jt^*x = jx$. Since $tjt = t$, we have $tjt^*x = tjx = x$.

Since $t^*x - x \in \ker j$, we have $V = \ker j \oplus \text{im } t^*$. If $jt^*x = 0$, then $jx = 0$, and consequently $x \in \ker j \cap \text{im } t = \{0\}$. Hence $V = \ker j \oplus \text{im } t^*$. Since $\text{im } s = t^* \text{im } t = \text{im } t^*$, we obtain $V = \ker j \oplus \text{im } s$.

Since

$$t^*x \cdot jt^*y = x \cdot jtjt^*y = x \cdot jy \quad \text{for all } x, y \in \text{im } t,$$

$$\text{we have } su \cdot jsv = t^*tu \cdot jt^*tv = tu \cdot jtv = t^*tu \cdot v = su \cdot v \quad \text{for all } u, v \in V'.$$

\square

Proof of (c1). Take any $y \in V$. Since $\ker j \perp \operatorname{im} j$, we have $x_1 \cdot jy = 0 = x_1 \cdot jsjy$ for every $x_1 \in \ker j$. Also $x_2 \cdot jy = x_2 \cdot jsjy$ for every $x_2 \in \operatorname{im} s$. Since $V = \ker j \oplus \operatorname{im} s$, we have $x \cdot jy = x \cdot jsjy$ for every $x \in V$. Then by the unimodularity of $\cdot : V \times V' \rightarrow \mathbb{Z}$ we have $j = jsj$. \square

Proof of (c2). If $sjx = 0$, then $jsjx = 0$. So by (c1) $jx = 0$. Therefore $\operatorname{im} j \cap \ker s = 0$. Since $\operatorname{im} j$ is a direct summand, by rank considerations $V' = \operatorname{im} j \oplus \ker s$. \square

Proof of (c3). By (c2) $\operatorname{im} s = s \operatorname{im} j$. Also $\operatorname{im} j = j \operatorname{im} s$. So the abbreviations $j : \operatorname{im} s \rightarrow \operatorname{im} j$ and $s : \operatorname{im} j \rightarrow \operatorname{im} s$ are surjective. Hence

$$jsj = j \iff js|_{\operatorname{im} j} = \operatorname{id}(\operatorname{im} j) \iff sj|_{\operatorname{im} s} = \operatorname{id}(\operatorname{im} s) \iff sjs = s.$$

\square

Proof of (c4). We have $jtj = jsj = j$ by (c1). We have $\operatorname{im} t \supset tj(V) = sj(V) = \operatorname{im} s$ by (c2). Hence $V = \ker j \oplus \operatorname{im} t$ and $tj|_{\operatorname{im} s} = sj|_{\operatorname{im} s} = \operatorname{id}(\operatorname{im} s)$ by (c3). \square

Proof of the Pre-elementary class Lemma 4.12. Since $H_3(\partial W)$ is free abelian, $\operatorname{im} j$ is a direct summand in $H_4(W, \partial)$. Apply Lemma 4.14.ab to $V = H_4(W)$, $V' = H_4(W, \partial)$, $\cdot = \cap_{\partial}$ and $j = j_W$. We obtain a 2-homomorphism s . Let us show how to modify (W, z, s) to achieve property (3) from the definition of a pre-elementary class.

Since $\rho_d \partial_W z^2 = 0$, $\rho_d z^2 \in \rho_d \operatorname{im} j_W$. Hence there are $a \in H_4(W)$ and $b \in H_4(W, \partial)$ such that $z^2 = j_W a + db$. So $\rho_d j_W s z^2 = \rho_d j_W s j_W a \stackrel{(2)}{=} \rho_d j_W a = \rho_d z^2$. Here (2) holds by Lemma 4.14.c1. Since the residues in the Pre-elementary class Lemma 4.12 are independent of the choice of $\overline{z^2} \in j_W^{-1} \rho_d z^2$, we may take $\overline{z^2} := \rho_d s z^2$.

Below we prove that

- (a) we can change $\eta(z, s) := sz^2 \cap_{\partial} (z^2 - p_W^*)$ by $2d$ without changing $sz^2 \cap_{\partial} z^2$;
- (b) we can simultaneously change $\eta(W, z, s)$ by $d^2 - d$ and $sz^2 \cap_{\partial} z^2$ by d^2 .

If d is odd, applying (b) we make $sz^2 \cap_{\partial} z^2$ even keeping $\eta(z, s)$ divisible by $\widehat{d} = \gcd(d, 3)$. Then applying (a) we can change $\eta(z, s)$ by $2d$ keeping $sz^2 \cap_{\partial} z^2$ even.

If d is even, $sz^2 \cap_{\partial} z^2$ is even by the hypothesis. Applying (a,b) we can change $\eta(z, s)$ by $\gcd(2d, d^2 - d) = d$ keeping $sz^2 \cap_{\partial} z^2$ even.

Take $(S^2)^4$ and the class z_S which is the sum of four summands, each represented by a product of three 2-spheres and a point. Then $z_S^4 = 24$. Since $(S^2)^4$ is almost parallelizable, we have $p_{(S^2)^4}^* = 0$. Taking connected sums with copies of $((S^2)^4, z_S)$ we can change $\eta(z, s)$ by any multiple of 24 while keeping $sz^2 \cap_{\partial} z^2$ even.

So we obtain that $\eta(z, s) = 0$ and $sz^2 \cap_{\partial} z^2$ is even.

By [KS91, spin case of (2.4) and Proposition 2.5] there is a closed spin 8-manifold W_0 and $z_0 \in H_6(W_0)$ such that $z_0^2 \cap_{W_0} (z_0^2 - p_{W_0}^*) = 0$ and $z_0^2 \cap_{W_0} z_0^2 = 2$. Taking connected sums with copies of (W_0, z_0) we can change $sz^2 \cap_{\partial} z^2$ by any multiple of 2 without changing $\eta(W, z, s)$. So we can obtain $sz^2 \cap_{\partial} z^2 = 0$ while keeping $\eta(z, s) = 0$.

Let $\mathbb{H}P^2$ be quaternionic projective space oriented so that its signature is given by $\sigma(\mathbb{H}P^2) = 1$. Recall that $\mathbb{H}P^2$ is 3-connected and $(p_{\mathbb{H}P^2}^*)^2 = 1$ [Mi56, Lemmas 3 and 4]. There is a 3-connected parallelizable 8-manifold \overline{E}_8 whose boundary is a homotopy sphere and whose signature is 8. Then $p_{\overline{E}_8}^* = 0$.

Since ∂W is parallelizable, $\partial_W p_W^* = 0$. By [CS11, Lemma 2.11.b] sp_W^* is a characteristic element for $\cap_W|_{\operatorname{im} s}$. Hence by Lemma 4.15.a $\sigma(W) = \sigma(\cap_W|_{\operatorname{im} s}) \equiv_{\text{mod } 8} sp_W^* \cap_W sp_W^*$. Therefore taking connected sums with copies of $\mathbb{H}P^2$ and \overline{E}_8 we can achieve $\sigma(W) = sp_W^* \cap_{\partial} p_W^* = 0$ while keeping $\eta(z, s) = sz^2 \cap_{\partial} z^2 = 0$.²³ \square

²³This covers a minor gap in [CS11, §4]: there we needed additionally to take connected sums with the \overline{E}_8 -manifold to kill α_W , and so ∂W will in general be changed by connected sum with a homotopy sphere.

Proof of (b). Denote $W_1 := W \# \mathbb{H}P^2 \# (-\mathbb{H}P^2)$. We have $H_4(\mathbb{H}P^2 \# (-\mathbb{H}P^2)) \cong \mathbb{Z}^2$ with evident basis. In this basis the intersection form is $\text{diag}(1, -1)$ and $p_{\mathbb{H}P^2 \# (-\mathbb{H}P^2)} = (1, 1)$. Let z_1 be the preimage of z under the ‘connected sum’ isomorphism $H_6(W_1, \partial) \rightarrow H_6(W, \partial)$. In order to construct the new s (this is t^*t not s_1 , both defined below) let us define the lower two lines of the following diagram:

$$\begin{array}{ccccccc}
& & & \xrightarrow{ct} & & & \\
& & \xrightarrow{c\partial} & & \xrightarrow{s' := (s \oplus \partial) \oplus \text{id}} & & \xrightarrow{t' := i \oplus (t'' \oplus \text{id})} \\
\left(\begin{smallmatrix} z_1^2 \\ p_{W_1}^* \end{smallmatrix} \right) & \xrightarrow{c\partial} & \left(\begin{smallmatrix} z^2, 0, 0 \\ p_{W_1}^*, 1, 1 \end{smallmatrix} \right) & \xrightarrow{s' := (s \oplus \partial) \oplus \text{id}} & \left(\begin{smallmatrix} sz^2, \partial_W z^2, 0, 0 \\ sp_{W_1}^*, 0, 1, 1 \end{smallmatrix} \right) & \xrightarrow{t' := i \oplus (t'' \oplus \text{id})} & \left(\begin{smallmatrix} sz^2, 0, d \\ sp_{W_1}^*, 1, 1 \end{smallmatrix} \right) \\
\downarrow \in & & \downarrow \in & & \downarrow \in & & \downarrow \in \\
H_4(W_1, \partial) & \xrightarrow{c\partial, \cong} & H_4(W, \partial) \oplus \mathbb{Z}^2 & \xrightarrow{s', \cong} & \text{im } s \oplus H_3(\partial W) \oplus \mathbb{Z}^2 & \xrightarrow{t'} & H_4(W) \oplus \mathbb{Z}^2 \\
& \searrow t & & \dashrightarrow s \oplus \text{id} (\neq t' s') & & \dashrightarrow & \\
& & s_1, t^*t & \dashrightarrow & H_4(W_1) & \xrightarrow{c, \cong} &
\end{array}$$

Let c_∂ and c be the ‘connected sum’ orthogonal isomorphisms (for the form $\text{diag}(1, -1)$ on \mathbb{Z}^2). Let $\text{id} := \text{id}_{\mathbb{Z}^2}$. Let $s'(u, a, b) := s(u) \oplus \partial u \oplus (a, b)$. Since $H_3(W) = 0$, by Lemma 4.14.c2 $s \oplus \partial: H_4(W, \partial) \rightarrow \text{im } s \oplus H_3(\partial W)$ is an isomorphism. Hence s' is an isomorphism.

Since $H_3(\partial W)$ is free abelian and $d = \text{div}(\partial_W z^2)$, there is a map

$$t'': H_3(\partial W) \rightarrow \mathbb{Z}^2 \quad \text{such that} \quad t''(\partial_W z^2) = (0, d).$$

Let $t'(u, v, a, b) := u \oplus (t''(v) + (a, b))$. Let

$$V := H_4(W_1), \quad V' := H_4(W_1, \partial), \quad \cdot := \cap, \quad j := j_{W_1}, \quad s_1 := c^{-1}(s \oplus \text{id})c_\partial, \quad t := c^{-1}t's'c_\partial,$$

so that the undashed lines of the diagram commute. Then $c \text{im } s_1 = \text{im } s \oplus \mathbb{Z}^2 = \text{im } t' = c \text{im } t$. Clearly, s_1 is a 2-homomorphism. Also

$$tj = c^{-1}t's'c_\partial j = c^{-1}t's'(j_W \oplus \text{id})c = c^{-1}t'((sj_W \oplus 0) \oplus \text{id})c = c^{-1}s(j_W \oplus \text{id})c = c^{-1}(s \oplus \text{id})c_\partial j = s_1 j.$$

Hence by Lemma 4.14.b,c4 for s_1 we obtain that t^*t is a 2-homomorphism.

For each $u_1, u_2 \in H_4(W_1, \partial)$ by definition of t^* we have

$$t^*tu_1 \cap_{W_1} u_2 = tu_1 \cap_{W_1} tu_2 = v_1 \cap_W v_2 + a_1 a_2 - b_1 b_2, \quad \text{where} \quad (v_k, a_k, b_k) = ct u_k.$$

Clearly, the images of z_1^2 are as shown in the first line of the diagram. Since ∂W is parallelizable, the images of $p_{W_1}^*$ are as shown in the first line of the diagram. Hence

$$t^*tz_1^2 \cap_{W_1} z_1^2 = sz^2 \cap_W sz^2 - d^2 \quad \text{and} \quad \eta(z_1, t^*t) - \eta(z, s) = 0 \cdot (0 - 1) - d \cdot (d - 1) = -d^2 + d.$$

□

Proof of (a). By [Mi56] there is a D^4 -bundle over S^4 whose Euler class is 0 and whose first Pontryagin class is 4. The double of this bundle is an S^4 -bundle $S^4 \tilde{\times} S^4$ over S^4 whose first Pontryagin class is 4. We have $H_4(S^4 \tilde{\times} S^4) \cong \mathbb{Z}^2$ with evident basis. In this basis $p_{S^4 \tilde{\times} S^4} = (2, 0)$ and the intersection form is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Analogously to the proof of (b) with $\mathbb{H}P^2 \# (-\mathbb{H}P^2)$ replaced by $S^4 \tilde{\times} S^4$ we construct W_1, z_1 and t^*t . Then for each $u_1, u_2 \in H_4(W_1, \partial)$ we have

$$t^*tu_1 \cap_{W_1} u_2 = tu_1 \cap_{W_1} tu_2 = v_1 \cap_W v_2 + a_1 b_2 + a_2 b_1, \quad \text{where} \quad (v_k, a_k, b_k) = ct u_k.$$

Also $ct \begin{pmatrix} z_1^2 \\ p_{W_1}^* \end{pmatrix} = \begin{pmatrix} sz^2, 0, d \\ sp_{W_1}^*, 2, 0 \end{pmatrix}$. Then $t^*tz_1^2 \cap_{W_1} z_1^2 = sz^2 \cap_W sz^2$ and $\eta(z_1, t^*t) = \eta(z, s) - 2d$. □

4.7 Proof of the Elementary pair Lemma 4.13

Lemma 4.15. *Let W be an 8-manifold satisfying the assumptions $(*)$ and $(**)$ of the Elementary pair Lemma 4.13. Let $s: H_4(W, \partial) \rightarrow H_4(W)$ be a homomorphism such that $H_4(W) = \text{im } i_W \oplus \text{im } s$ (additively which implies orthogonally w.r.t. \cap_W). Denote $j_k := j_{M_k, W}$ and $S := \text{im } s$. Denote by the superscript \perp the orthogonal complement with respect to \cap_{01} , unless another intersection product is indicated as subscript. Then*

- (a) S is free abelian and the form $\cap_W|_S$ is unimodular;
- (b) $j_0|_S, j_1|_S$ are injective, $H_4(W, M_k) = (j_{1-k}S)^\perp \oplus j_kS$, $k = 0, 1$, and the restrictions of \cap_{01} both to $j_0S \times j_1S$ and to $(j_1S)^\perp \times (j_0S)^\perp$ are unimodular;
- (c) $j_k \text{im } i_W$ is a half-rank direct summand in the free abelian group $(j_{1-k}S)^\perp$;
- (d) if

$$a \in H_4(W, \partial), \quad q \in \ker j_{M_0, \partial W} \subset H_4(\partial W) \quad \text{and} \quad q \cap_{\partial W} \partial_W a = \text{div}(\partial_W a),$$

then there is a subgroup $U \subset \text{im } i_W$ such that $U \cap_{\partial} a = 0$ and the pair $j_k|_U: U \rightarrow (j_{1-k}S)^\perp$, $k = 0, 1$, is elementary.

Proof of (a). Since the torsion of $H_4(W)$ is contained in $\text{im } i_W = H_4(W)_{\cap_W}^\perp$, the group S is free abelian. Since $\cap_{\partial}: H_4(W) \times H_4(W, \partial) \rightarrow \mathbb{Z}$ is unimodular, $x \cap_W y = x \cap_{\partial} j_W y$ for all $x, y \in H_4(W)$ and $H_3(\partial W)$ is free abelian, it follows that the form $\cap_W|_S$ is unimodular. \square

Proof of (b). Since $j_0 x \cap_{01} j_1 y = x \cap_W y$ for all $x, y \in H_4(W)$, it follows that $\cap_{01}|_{j_0 S \times j_1 S}$ is unimodular. Then $j_0|_S$ and $j_1|_S$ are injective. So if $x \in S$ and $j_k x \cap_{01} j_{1-k} S = 0$, then $x = 0$. Also, for each $y \in H_4(W, M_k)$ the \cap_{01} -intersection with y defines a linear map $j_{1-k} S \rightarrow \mathbb{Z}$. Hence there is a class $y_S \in j_k S$ such that $y \cap_{01} x = y_S \cap_{01} x$ for each $x \in j_{1-k} S$. Then $y = y_S + (y - y_S)$ and $(y - y_S) \cap_{01} j_{1-k} S = 0$. Thus $H_4(W, M_k) = (j_{1-k} S)^\perp \oplus j_k S$, $k = 0, 1$. Since \cap_{01} and $\cap_{01}|_{j_0 S \times j_1 S}$ are unimodular, $\cap_{01}|_{(j_1 S)^\perp \times (j_0 S)^\perp}$ is unimodular. \square

Some notation for the proofs of Lemmas 4.15.c,d and 4.13. Denote by $i_{W,k}, \partial_{W,k}$ homomorphisms from the exact sequence of the triple $(W, \partial W, M_k)$. Denote by i_k, j_k, ∂_k and $\tilde{i}_k, \tilde{j}_k, \tilde{\partial}_k$ the homomorphisms from the exact sequences of the pairs (W, M_k) and $(\partial W, M_k)$, respectively. Recall that $H_4(W)_{\cap_W}^\perp = \text{im } i_W$. Consider the following diagram:

$$\begin{array}{ccccccc}
 & & H_5(W, \partial) = 0 & & & & H_3(W) = 0 \\
 & & \downarrow \partial_W & \searrow \partial_{W,k} & & & \nearrow i_k \\
 H_4(M_k) & \xrightarrow{\tilde{i}_k} & H_4(\partial W) & \xrightarrow{\tilde{j}_k} & H_4(\partial W, M_k) & \xrightarrow{\tilde{\partial}_k=0} & H_3(M_k) \\
 & \searrow i_k & \downarrow i_W & & \downarrow i_{W,k} & \nearrow \partial_k & \\
 & & H_4(W) & \xrightarrow{j_k} & H_4(W, M_k) & & \\
 & & \parallel & & \parallel & & \\
 & & \text{im } i_W \oplus S & \xrightarrow{j_k} & (j_{1-k} S)^\perp \oplus j_k S & &
 \end{array}$$

Proof of (c). We have

$$\frac{(j_{1-k} S)^\perp}{j_k \text{im } i_W} \stackrel{(1)}{\cong} \text{coker } j_k \stackrel{(2)}{\cong} H_3(M_k) \stackrel{(3)}{\cong} H_4(\partial W, M_k) \stackrel{(4)}{\cong} \text{im } i_{W,k} \stackrel{(5)}{=} j_k \text{im } i_W.$$

Here

- (1) is obtained by adding $j_k S$ both to nominator and denominator and using (b);
- (2) holds because $H_3(W) = 0$, hence ∂_k is an epimorphism;

• (3) holds by Poincaré-Lefschetz duality because both $H_3(M_k)$ and $H_4(\partial W, M_k) \cong^{\text{ex}} H_4(M_{1-k}, \partial)$ are free abelian;

• (4) holds because $H_5(W, \partial) = 0$, hence $i_{W,k}$ is injective;

• (5) holds because \tilde{j}_k is surjective.

Since $H_3(M_k)$ is free abelian, $j_k \text{ im } i_W \cong \frac{(j_{1-k}S)^\perp}{j_k \text{ im } i_W}$ is free abelian. This implies (c). \square

Proof of (d). Since M_k is torsion free, by Poincaré-Lefschetz duality $H_4(\partial W, M_0) \cong^{\text{ex}} H_4(M_1, \partial)$ is free abelian. Since \tilde{j}_0 is surjective, it follows that there is a subgroup

$$U'' \subset H_4(\partial W) \quad \text{such that} \quad \tilde{j}_0|_{U''} : U'' \rightarrow H_4(\partial W, M_0) \quad \text{is an isomorphism.}$$

Since $H_3(\partial W)$ is free abelian, there is a class

$$a_0 \in H_3(\partial W) \quad \text{such that} \quad \partial_W a = a_0 \text{ div}(\partial_W a).$$

$$\text{Define } U' := \{u - (a_0 \cap_{\partial W} u)q : u \in U''\} \quad \text{and} \quad U := i_W U'.$$

Since $q \cap_{\partial W} \partial_W a = \text{div}(\partial_W a)$, we have $\partial_W a \cap_{\partial W} U' = 0$. Thus $U \cap_{\partial} a = 0$. So by (c) it remains to prove that $j_k|_U$ is an isomorphism onto $j_k \text{ im } i_W$.

Since $\ker \tilde{j}_0 = \ker \tilde{j}_1$, the map $\tilde{j}_1|_{U''}$ is injective. Then $U'' \cap_{\partial W} \text{im } \tilde{i}_k = 0$. This and the fact that $q \in \text{im } \tilde{i}_0 = \text{im } \tilde{i}_1 = \ker \tilde{j}_0 = \ker \tilde{j}_1$ imply that $U' \cap_{\partial W} \text{im } \tilde{i}_0 = 0$. Since $\ker \tilde{j}_0 = \ker \tilde{j}_1$, we have $\text{im } \tilde{i}_0 = \text{im } \tilde{i}_1$. Therefore $U' \cap_{\partial W} \text{im } \tilde{i}_1 = 0$. Thus $\tilde{j}_k|_{U'}$ is injective. Since $H_5(W, \partial) = 0$, the map i_W is injective. Hence $U \cap_W i_W \ker \tilde{j}_k = 0$. We have $i_W \ker \tilde{j}_k = i_W \text{im } \tilde{i}_k = \text{im } i_k = \ker j_k$. Thus $j_k|_U$ is injective.

Since $\ker \tilde{j}_0 = \ker \tilde{j}_1$, we have $H_4(\partial W) = U'' + \ker \tilde{j}_0 = U'' + \ker \tilde{j}_1$. Since $q \in \ker \tilde{j}_0 = \ker \tilde{j}_1$, it follows that $H_4(\partial W) = U' + \ker \tilde{j}_0 = U' + \ker \tilde{j}_1$. So $\tilde{j}_1 U' = \tilde{j}_1 H_4(\partial W)$. Therefore we have $j_k U = j_k i_W U' = i_{W,k} \tilde{j}_k U' = i_{W,k} \tilde{j}_k H_4(\partial W) = j_k \text{ im } i_W$. \square

Proof of the Elementary pair Lemma 4.13. The group $H_4(W, M_k)$ is torsion free for $k = 0, 1$. (Indeed, consider the Poincaré dual of the exact sequence of the pair (W, M_{3-k}) :

$$H_5(W, \partial) \rightarrow H_4(M_{3-k}, \partial) \rightarrow H_4(W, M_k) \rightarrow H_4(W, \partial).$$

By the assumptions

$$H_5(W, \partial) = 0, \quad \text{Tors } H_4(M_{3-k}, \partial) = \text{Tors } H_2(M_{3-k}) = 0 \quad \text{and} \quad \text{Tors } H_4(W, \partial) = \text{Tors } H_3(W) = 0.$$

Hence $H_4(W, M_k)$ is torsion free.)

Since z is pre-elementary, there is a homomorphism s from the definition of a pre-elementary class. Denote $S := \text{im } s$. Let

$$\widehat{U} := \{u \in S \mid lu = msz^2 + nsp_W^* \text{ for some integers } l, m, n\}.$$

Since z is pre-elementary, $\widehat{U} \cap_W \widehat{U} = 0$. By Lemma 4.15.a S is free abelian and the form $\cap_W|_S$ is unimodular. Then there is a subgroup

$$T \subset S \quad \text{such that} \quad \widehat{U} \subset T, \quad \text{rk } T = 2 \text{rk } \widehat{U} \quad \text{and} \quad \cap_W|_T \text{ is unimodular.}$$

Hence $\sigma(T) = 0$. Since both $\cap_W|_S$ and $\cap_W|_T$ are unimodular, $T \cap T_{\cap_W}^\perp = 0$ and $\text{rk } T_{\cap_W}^\perp = \text{rk } S - \text{rk } T$, we have $S = T \oplus T_{\cap_W}^\perp$. So $\sigma(T_{\cap_W}^\perp) = \sigma(W) - \sigma(T) = 0$. Hence there is a half-rank direct summand

$$\widetilde{U} \subset T_{\cap_W}^\perp \quad \text{such that} \quad \widetilde{U} \cap_W \widetilde{U} = 0.$$

Let $U_S := \widehat{U} \oplus \widetilde{U}$.

We have $U_S \cap_{\partial} z^2 = U_S \cap_{\partial} p_W^* = 0$ and the pair $j_k|_{U_S}: U_S \rightarrow j_k S$, $k = 0, 1$, is elementary.

(Indeed, since z is pre-elementary, $\widehat{U} \cap_{\partial} z^2 = \widehat{U} \cap_{\partial} p_W^* = 0$. Also $\widetilde{U} \cap_W \widehat{U} = 0$. Hence by the properties (2) and (3) of s we obtain $U_S \cap_{\partial} z^2 = U_S \cap_{\partial} p_W^* = 0$. Since $\widehat{U} \cap_W \widetilde{U} = 0 = \widetilde{U} \cap_W \widehat{U}$, we have $j_0 U_S \cap_{01} j_1 U_S = 0$. By Lemma 4.15.b $j_k|_S$ is injective. Since \widehat{U} and \widetilde{U} are half-rank direct summands in T and in $T_{\cap_W}^{\perp}$, respectively, U_S is a half-rank direct summand in S . So $j_k U_S$ is a half-rank direct summand in $j_k S$.)

Applying Lemma 4.15.d to $a = z^2$ we obtain a subgroup $U_{\partial} \subset \text{im } i_W$. Since ∂W is parallelizable, $p_1(\partial W) = 0$. Hence $\text{im } i_W \cap_{\partial} p_W^* = 0$. Therefore $U := U_S \oplus U_{\partial}$ is as required. \square

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